A behavioral theory of the consumer under reference dependent preferences and certainty.*

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Abstract

We extend classical consumer theory to account for reference dependence and loss aversion under complete certainty. The classical results obtain as a special case. Several new results emerge—there is a kink in the demand curve at the reference point, consumers are subject to money illusion, and some kinds of inconsistencies of preferences can be accounted for. However, the reference dependent model does not predict the symmetry of the Hicksian cross partial derivatives, nor the downward slope of Hicksian curves (so the compensated law of demand does not survive), nor the concavity of the expenditure function. We show how several classical results can be modified to take account of reference dependence; these include Shepherd’s Lemma, Roy’s identity, Slutsky equation, and welfare measures such as equivalent and compensating variation. Finally, the classical model underestimates the dead-weight loss from taxation; simulations show that this underestimation is by several orders of magnitude.

*This is an extremely preliminary version of the paper. Not for circulation.
1. Introduction

There is robust evidence from psychology and biology of reference dependence in the response to external stimuli for all forms of animal and plant life to changes in the environment. Evidence indicates that when exposed to external stimuli, such as temperature, brightness, and pain, individuals are more sensitive to changes rather than levels (Helson, 1964). In order to fix ideas on reference dependence, consider the following well known experiment: Put your right hand in cold water and your left hand in warm water until both hands have had time to adjust to the temperature. Then, simultaneously take out both hands and dip them in lukewarm water. Although both hands experience the identical lukewarm temperature, the right hand feels warm and the left hand feels cold.

There is now wide ranging evidence from economics of reference dependent human decisions. This includes explanations of the equity premium puzzle, tax evasion puzzles, disposition effect, backward bending labor supply curve, endowment effect, asymmetric price elasticities, exchange asymmetries, choices of professional golf players, backward bending labour supply curve, role of initial contracts as reference points around which future renegotiation is structured, and goal setting choices of individuals (Kahneman and Tversky, 2000; Dhami, 2016). Furthermore, there is good neural evidence for reference dependence and loss aversion from the emerging literature in neuroeconomics (Glimcher and Fehr, 2014).

Reference points were introduced into economics with the ground breaking Nobel Prize winning work of Daniel Kahneman and Amos Tversky on prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992). Prospect theory is a theory of risk, uncertainty and ambiguity but also applies to choices under certainty. The status-quo turns out to be an important reference point and was emphasized in prospect theory (Kahneman and Tversky, 2000). This is not only eminently reasonable in contexts where risk and uncertainty play no essential role but it is supported by the evidence from choices made by humans and chimpanzees (Chen et al., 2006; Lakshminarayanan et al., 2011; Santos and Platt, 2014).

Status-quo need not be the only candidate for a reference point. Other possible choices for a reference point could be a fair outcome, or a legal/social entitlement; clearly the context dictates the appropriate reference point. The reference point could also be state-dependent wealth, average wealth, or desired wealth. Under risk, the reference point has been made endogenous by using the rational expectations of future wealth (Köszegi and Rabin, 2006, 2007). Indeed expectations of future income may serve as important reference points (Abeler et al., 2011; Falk and Knell, 2004). However, our focus is on choice under certainty.

If the outcome turns out to be better (respectively worse) as compared to the reference
point, then the decision maker is said to be in the domain of gains (respectively losses). Experimental and field evidence suggests that utility is evaluated differently in the domain of gains and losses. In particular, individuals are loss averse in the domain of losses—losses bite, on average, about 2.5 times equivalent gains although there is substantial variation in individual choice and loss aversion is context/frame dependent (Kahneman and Tversky, 2000). Thus, it is not simply a matter of introducing a reference point in classical consumer theory (which it is eminently capable of accommodating) but rather a matter of specifying exactly how human behavior is different in the domain of gains and losses.

1.1. A brief description of the paper and results

We pose the following question: Under certainty, how are the results in classical consumer theory altered in the presence of reference dependence and loss aversion, and what important empirical findings can be addressed using this framework. Our aim is to provide a fairly conservative behavioral analogue of, say, the first 100 pages of the classic text by Mas-Colell et al. (1995). Classical consumer theory is a special case of our framework when there is no reference dependence. We extend classical preferences to allow for additive gain-loss utility in the spirit of Thaler (2008), Köszegi and Rabin (2006, 2007), and Heidhues and Köszegi (2008).

Our main findings are as follows. First, there is a kink in the demand curve at the reference point. Second individuals are subject to money illusion. Third, the symmetry of the Hicksian cross partial derivatives cannot be used as an identification restriction in empirical work. Fourth, the classical measures of welfare loss such as equivalent variation and compensating variation no longer provide the correct measure of welfare change; we provide the necessary modification. Fifth, the compensated law of demand does not survive, so Hicksian demand curves may no longer be downward sloping. Most classical results, such as concavity of the expenditure function, Shepherd’s Lemma, Roy’s identity and the Slutsky equation do not hold in their classical form; we propose the necessary modifications. Sixth, the classical framework underestimates the deadweight loss from taxation; our simulations show that this underestimation is significant. All the classical results can be recovered as a special case of our model that has no reference dependence.

1.2. Empirical evidence of reference dependence in consumer choice

Thaler (2008) makes a distinction between transaction utility and acquisition utility in purchase decisions.\(^1\) Suppose that an individual has a valuation for an object, \(v > 0\), and pays a price \(p > 0\) to acquire it. Then, the net valuation of the object, or the acquisition

utility, is $U_A = v - p$; this corresponds to consumer surplus. A related concept is transaction utility, $U_T$, defined as follows: $U_T(p, p) = \bar{p} - p$, where $\bar{p}$ is the reference price or the regular price that the consumer expects to pay. Thaler (2008) suggested that the most important determinant of $\bar{p}$ is fairness, which depends on the costs of the seller. This is supported by the experimental evidence (Thaler, 2008, Table 1, p. 19). Transaction utility would dictate buying the good only if the actual price is below the reference price, i.e., $p \geq \bar{p}$. The total utility from the purchase is $U(v, p, p) = U_A + U_T$.

**Example 1** (Thaler, 2008): You are lying on the beach on a hot day. All you have to drink is ice water. For the last hour you have been thinking about how much you would enjoy a nice cold bottle of your favorite brand of beer. A companion gets up to go make a phone call and offers to bring back a beer from the only nearby place where beer is sold (a fancy resort hotel) [a small, run-down grocery store]. He says that the beer might be expensive and so asks how much you are willing to pay for the beer. He says that he will buy the beer if it costs as much or less than the price you state. But if it costs more than the price you state he will not buy it. You trust your friend, and there is no possibility of bargaining with the (bartender) [store owner]. What price do you tell him?

The median willingness to pay for the beer from the resort and the grocery store was, respectively, $2.65$ and $1.50$ (in 1984 prices). To rationalize this, suppose that the decision criterion of the consumer is to purchase the object if $U_T \geq 0$. Hence, if the consumer has a higher reference price, $\bar{p}$, for the resort relative to the grocery shop, then he would also be willing to pay a higher actual price, $p$. These results are quite striking because in each case, the object of consumption is an identical bottle of beer in an identical environment and there are no strategic considerations so acquisition utility $U_A$ is likely to be identical.

Consider another example:

**Example 2** (Thaler, 1999, p.184): A friend of mine was once shopping for a quilted bedspread. She went to a department store and was pleased to find a model she liked on sale. The spreads came in three sizes: double, queen and king. The usual prices for these quilts were $200$, $250$ and $300$ respectively, but during the sale they were all priced at only $150$. My friend bought the king-size quilt and was quite pleased with her purchase, though the quilt did hang a bit over the sides of her double bed.

The concepts of transactions and acquisition utility would also appear to explain why we sometimes buy items at a sale that we do not find very useful subsequently. The reason is that transaction utility is salient at the time of a purchase. This rule, namely, purchase when the actual price is lower than the reference price can also be used to explain the puzzle in Example 2, where the individual buys a king size quilt when a queen size quilt
would have been more appropriate. Transaction utility corresponds to our desire to get a good deal; indeed it is the extra utility that appears to arise from a good deal.

Muehlbacher et al. (2011) induce a reference point for a product and then elicit the desire to buy the product at different prices that are above or below the reference price. They find that there is an asymmetric response of the desire to buy; when prices are above the reference price, the desire to buy is more severely curtailed relative to the increase in demand when the price is below the reference price. Furthermore, they vary the acquisition value of the product (quality of a pair of trousers) and find that changes in acquisition utility do not moderate the impact of transaction utility.

Intriguing results indicate that transaction utility may enhance acquisition utility (Gre- wal et al., 1998); a good deal on a product may bring to mind a disproportionate number of reasons for the usefulness of the product. The salience of transaction utility relative to acquisition utility had been termed as utility blindness by Liu (2014) and coupon-proneness by Lichtenstein et al. (1999). This has been justified on the grounds of cognitive load on consumers as they perform the necessary utility calculations in buying products under time pressure and the cognitive load from memorizing grocery lists (Liu, 2014). The effectiveness of transaction utility in facilitating purchases through the use of coupons that reduce the price of the good is documented by Lichtenstein et al. (1999) and Milkman and Beshears (2009).

Some of the earliest experiments that demonstrated fairness concerns of consumers for prices were conducted in the mid 1980’s by Daniel Kahneman, Jack Knetsch and Richard Thaler, based on telephone surveys in Canada.²

Example 3: Kahneman et al. (1986) asked people if they would view a hardware store’s action to increase the price of shovels from $15 to $20 following a snowstorm as fair? Most people (82%) thought that such a price increase was unfair. However, they did not perceive an increase in price following an increase in the cost of inputs as unfair. In actual practice, many firms typically go to great lengths to establish their credentials for fair behavior.

Consider the following survey questions from Kahneman et al. (1991):

Example 4: Question 1a. "A shortage has developed for a popular model of automobile, and customers must now wait two months for delivery. A dealer has been selling these cars at list price. Now the dealer prices this model at $200 above list price." Out of 130 respondents, 29% found the behavior of dealers acceptable, while 71% found it unacceptable.

²See the interesting background to these surveys in Thaler (2015, Ch. 14) and a discussion of the other experiments that they conducted.
Question lb. "A shortage has developed for a popular model of automobile, and customers must now wait two months for delivery. A dealer has been selling these cars at a discount of $200 below list price. Now the dealer sells this model only at list price." Out of 123 respondents, 58% found the behavior of dealers acceptable while 42% found it unacceptable.

The increase in price is identical in the two cases. However, the two economic environments in Q1a and Q1b generate different entitlements for both parties and a departure from the entitlement is considered unfair.

In other contexts, the reference price interacts with other aspects of the problem such as the attitudes to surcharges and opportunity costs. In particular, foregoing a discount is less aversive relative to a surcharge.

Example 5: Consider, for instance, charging for credit card usage in the early days of credit cards (Thaler, 2015). Initially, there was a 3% surcharge on the use of credit card payments in retail stores, which a consumer could avoid by paying cash. The credit card companies successfully lobbied the retail price of a $1 item to be raised to $1.03. So cash buyers could pay a discounted price of $1, while card users could pay the full price of $1.03. Card users thus do not pay a surcharge for their purchase. As Thaler puts it (pp. 18): “Paying a surcharge is out-of-pocket, whereas not receiving a discount is a "mere" opportunity cost.” Clearly direct out-of-pocket costs are more aversive relative to the more indirect concept of opportunity costs that is less well understood by many.

In the marketing literature, it is a widely accepted empirical regularity that consumers compare the actual price of a product with its internal reference price (IRP)—a subjective reference price—and such comparisons directly impart positive or negative utility (Kalyanaram and Winer, 1995). For a recent survey that echoes and updates these findings, see Mazumdar et al. (2005). There is inadequate research in the marketing literature on what constitutes a reference price for a consumer. Rajendran (2009) pits various theories of reference price against each other. He tests to see if the reference price is some subjective notion of a fair price, a reference price based on past purchases of the consumer, and a reference price based on a comparison of the price against the average price of competing brands. He finds that consumers are motivated mainly be what they believe is a fair price. However, the various components may be hard to separate in the absence of cleverly conducted experiments. For instance, observed past prices or the prices of competing brands may inform one’s perception of what is a fair price. The literature is aware of these issues (Thaler, 1985; Nagle and Holden, 1995; Rajendran, 2009).

If consumers are influenced by reference price considerations then they may respond asymmetrically to price increases and decreases relative to the current price that may be
perceived as a reference price. Thus, we may get asymmetric price elasticities for gains (prices below the reference price) and losses (prices higher than the reference price). Putler (1992) examined the weekly retail price changes for eggs in Southern California over the period 1981-83. Relative to the current price, he found that consumers were 2.5 times more responsive to price increases as compared to a price decrease. Thus, at any time, there was a kinked demand curve for eggs around the current price.

Mazumdar et al. (2005) give a wide ranging survey of the determinants of reference prices from a marketing perspective. They identify a large number of variables that play relatively little role in economics. Several empirical papers in marketing try to identify consumer behavior, choices among competing brands, and price promotions that are influenced by reference prices (Winer, 1986, 1988; Kalwani et al., 1990; Raman and Bass, 1988; Putler, 1992, Hardie et al., 1993; Kalwani and Yim, 1992; Raman and Bass, 2002).

1.3. Brief relation to existing literature

We are interested in extending classical consumer theory to take account of reference dependence and loss aversion, when there is no uncertainty and the consumer takes the economic environment (prices and income) as given. For the classical analogue in the absence of reference dependence, see the first 100 pages in Mas-Colell et al. (1995) or Chapter 1 in Jehle and Reny (2011). Several papers examine the pricing decisions of firms in the presence of loss averse consumers (Heidhues and Köszegi, 2008; Spiegler, 2012). Köszegi and Rabin (2006, 2007) develop the theory of the consumer under reference dependence when risk and uncertainty play a central role. These contributions are complementary to our work.

Following a formulation suggested by Sibly (2007), Aherns et al. (2014) incorporate reference price effects into a macroeconomic model. The utility function is treated as a CES function of the \( n \) goods in which the ‘price relative to a reference price’ directly enters the utility function. The power coefficient of this variable is then asymmetric in gains (price lower than reference price) and losses (price higher than the reference price). Their main focus is to show that the pricing behavior of monopolistic firms in response to temporary and permanent shocks is different and it is determined by the reference price effect.\(^3\)

There has been an enormous amount of empirical and theoretical work on reference dependent preferences and on loss aversion. Yet, none of the theoretical work deals with the issues that we raise, with the exception of Putler (1992). Putler (1992) argues that under reference dependence, demand curves may be kinked and money illusion in consumer choices may be present. However, it is often not clear if a formal proof of the claims has

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\(^3\) Aherns et al. (2014) also provide a nice list of references and a survey of reference price models.
been given. Furthermore, he considers only the utility maximization problem.

1.4. Schematic outline of the paper

Section 2 states the behavioral assumptions and explains the basic framework of analysis. Section 3 considers the utility maximization problem, the properties of the Marshallian demands, the properties of the indirect utility function, and the problem of apparent inconsistency of preferences. We also show that the demand curve is kinked at the reference price and the consumer is subject to money illusion. Section 4 considers the expenditure minimization problem and derives the behavioral analogues of Shepherd’s Lemma, Roy’s identity and the Slutsky equation. We also identify the transformed Hicksian demand and its properties, which plays a fundamental role in welfare analysis. Section 5 considers the problem of measuring consumer welfare when the economic environment changes. We consider the analogues of equivalent and compensating variation under reference dependence and also compare the deadweight loss of taxation with and without reference dependence.

2. Behavioral assumptions in the two goods case

Consider two goods \( x = (x_1, x_2) \) and a convex set \( X \subset \mathbb{R}_+^2 \) over which choices are made. The consumer’s exogenously given income is \( w > 0 \). We assume that the classical binary preference relation \( \succeq \) on \( X \), in the absence of reference dependence, is complete, transitive, and continuous.

**Definition 1**: The preference relation \( \succeq \) on \( X \) has the following properties:

(i) Completeness: If \( x, y \in X \) then either \( x \succeq y \), \( y \succeq x \) or both.

(ii) Transitivity: Let \( x, y, z \in X \). If \( x \succeq y \) and \( y \succeq z \), then \( x \succeq z \).

(iii) Continuity: For any sequence \( \{(x^n, y^n)\} \) such that for all \( n \), \( x^n, y^n \in X \), if \( x^n \succeq y^n \), \( (x^n, y^n) \to (x, y) \in X \) then \( x \succeq y \).

The preferences \( \succeq \) are *rational* if they satisfy completeness and transitivity. In conjunction with continuity, rationality ensures that preferences can be represented by a utility function \( u : X \to \mathbb{R} \). We assume \( u \) to be twice continuously differentiable. In addition, we assume that \( \succeq \) is (1) strongly monotone (hence, it is also locally non-satiated), and (2) convex. We note these definitions for further reference.

**Definition 2**: (1) (Strong monotonicity) Let \( x, y \in X \). If \( y \succeq x \) and \( y \neq x \) then \( y \succ x \).

(2) (Local non-satiation) Let \( x \in X \) be an arbitrary consumption bundle. Then for any \( \varepsilon > 0 \) there exists \( y \in X \), \( \|y - x\| < \varepsilon \), such that \( y \succ x \).

**Definition 3** (Convexity of \( \succeq \)): \( \succeq \) is convex on \( X \) if the set \( \{y \in X : y \succeq x \} \) is convex, i.e., for every \( x, y, z \in X \), such that \( y \succeq x \) and \( z \succeq x \), \( \alpha y + (1 - \alpha)z \succeq x \) for \( \alpha \in [0, 1] \).
A simple calculation shows that convexity of $\succeq$ implies that $u$ is quasiconcave.

**Definition 4**: (Quasiconcavity) Two alternative definitions of quasiconcavity are:

1. $u$ is quasiconcave if for distinct $x, y, z \in X$, such that $u(y) \geq u(x)$ and $u(z) \geq u(x)$ we have $\alpha u(y) + (1 - \alpha)u(z) \geq u(x)$ for $\alpha \in [0, 1]$.
2. $u$ is strictly quasiconcave if for distinct $x, y \in X$ and $\alpha \in (0, 1)$, $u(\alpha x + (1 - \alpha)y) > Min\{u(x), u(y)\}$.

Let us now extend the classical preferences to allow for reference dependence.

Suppose that the exogenously given reference price vector is $\bar{p} = (\bar{p}_1, \bar{p}_2)$; following the status-quo justification for reference points, we will sometimes find it useful to let the initial price vector be the reference price. $\bar{p}$ could also be some subjective notion of a ‘fair price’ or a price that consumers feel that they are ‘entitled to’ or some statistic of the past levels of prices that is encoded by consumers as the reference price. We remain agnostic about the exact source of the reference price. There is no uncertainty about the actual price vector $p = (p_1, p_2)$.

We extend the classical preferences $\succeq$ on $X$ to the preferences $\succeq_{\bar{p}, p}$ on $X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$. We assume that the preferences $\succeq_{\bar{p}, p}$ are complete, transitive, and continuous (see Definition 1). Hence, they can be represented by a continuous utility function $U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}$.

So if $x, y \in X$ then $x \succeq_{\bar{p}, p} y$ implies that at the reference price vector $\bar{p}$ and the actual price vector $p$, the consumption bundle $x$ is at least as good as the consumption bundle $y$. It may be the case that for another price vector $(p', p')$, and the corresponding preferences $\succeq_{p', p'}$, we have $y \succeq_{p', p'} x$. Classically the situation $x \succeq y$ and $y \succeq x$ cannot occur.

Another interpretation of these preferences is that each distinct vector $(\bar{p}, p)$ corresponds to a distinct context or a frame. Fixing a context/frame, preferences are rational because the preferences $\succeq_{\bar{p}, p}$ are complete, transitive, and continuous. Suppose that an observer does not take account of the context/frame dependence of preferences and erroneously believes that the consumer possesses classical preferences $\succeq$. Then, choices based on the correct underlying preferences $\succeq_{\bar{p}, p}$ in two different contexts such that $x \succeq_{\bar{p}, p} y$ and $y \succeq_{p', p'} x$ will lead the observer to erroneously conclude that preferences are inconsistent (i.e., $x \succeq y$ and $y \succeq x$). Our assumption of context dependent rationality of preferences is a major theme in Gintis (2009).

We assume that the utility representation of the preferences $\succeq_{\bar{p}, p}$ is given by

$$U(x, \bar{p}, p) = u(x_1, x_2) + \mu [g_1(\bar{p}_1, p_1)x_1 + g_2(\bar{p}_2, p_2)x_2], \mu \geq 0,$$

(2.1)

where the per unit gain/loss functions are given by

$$g_i(\bar{p}_i, p_i) = \begin{cases} (\bar{p}_i - p_i) & \text{if } \bar{p}_i \geq p_i \geq p_i \\ -\theta (p_i - \bar{p}_i) & \text{if } \bar{p}_i \leq p_i < p_i \end{cases}, i = 1, 2 \text{ and } \theta > 1.$$

(2.2)
The parameter $\theta$ represents loss aversion and $\mu$ is the relative weight on gain-loss utility. We will sometimes write

$$U(x, \bar{p}, p) = u(x_1, x_2) + \mu G(x, \bar{p}, p)$$

where $G(x, \bar{p}, p) = [g_1(\bar{p}_1, p_1)x_1 + g_2(\bar{p}_2, p_2)x_2]$ is total gain-loss utility. Our specification is fairly parsimonious because it introduces just two extra parameter $\mu$ and $\theta$. The special case $\mu = 0$ corresponds to the classical preferences, $\succeq$.

Our utility representation of preferences is motivated by Thaler (2008) and is similar to Köszegi and Rabin (2006, 2007) and Heidhues and Köszegi (2008). The main extension to Thaler (2008) is that we have additive separability in gain-loss utility across the goods. The main differences from the work of Köszegi and Rabin (2006, 2007) and Heidhues and Köszegi (2008) are twofold. First, there is no risk or uncertainty in our model. Second (and like Thaler, 2008), in our formulation, gain-loss utility is defined over prices but not quantities. In this respect, we are mainly motivated by the empirical evidence; there is persuasive empirical evidence about gain-loss utility in prices but, the best of our knowledge, not quantities. For future reference we note that

$$\frac{\partial g_i}{\partial p_i} = \begin{cases} -1 & \text{if } \bar{p}_i \geq p_i \\ -\theta & \text{if } \bar{p}_i < p_i \end{cases} \quad (2.3)$$

The expression in (2.3) will be a crucial determinant of kinks in important economic functions at the reference price.

**Notation:** We shall use the following notation in our diagrammatic illustrations. When the preferences are given by $\succeq_{\bar{p}, p}$ and a utility level $U$ is achieved at some consumption bundle, we shall write the utility as $U(\succeq_{\bar{p}, p})$. If the same level of utility $U$ is achieved at another set of preferences, $\succeq_{\bar{p}, p'}$, then we shall represent this utility as $U(\succeq_{\bar{p}, p'})$. By contrast, if the level of utility $U$ is achieved under classical preferences, $\succeq$, then we write it as $U(\succeq)$.

**3. The utility maximization problem (UMP)**

In the utility maximization problem (UMP), we have

$$x^* \in \arg\max_{x \in X} U(x, \bar{p}, p),$$

An even more nuanced gain-loss utility function could specify $G(x, \bar{p}, p) = a_1 g_1(\bar{p}_1, p_1)x_1 + a_2 g_2(\bar{p}_2, p_2)x_2$, where $a_i \geq 0$ and $a_1 + a_2 = 1$; $a_i$ represents that salience of gain-loss utility in the $i^{th}$ dimension. Such a strategy may be required in empirical work and our model can be easily extended to account for it. However, pedagogically we stick to the case $a_1 = a_2 = 1$. 
where $U$ is defined in (2.1), subject to the budget constraint
\[ B_{p,w} = \{ x \in X : p_1 x_1 + p_2 x_2 \leq w \}, \tag{3.1} \]
where $w > 0$ is the exogenously given income of the consumer, $\bar{p} = (\bar{p}_1, \bar{p}_2)$ is the reference price vector and $p = (p_1, p_2)$ is the actual price vector. Collectively, the vector $(\bar{p}, p, w)$ summarizes the economic environment faced by the consumer in making optimal consumption decisions. The non-negativity constraints are given by
\[ x_i \geq 0, i = 1, 2. \tag{3.2} \]

Let $U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$ represent continuous, differentiable, locally non-satiated preferences, $\succeq_{\bar{p},p}$, and let $p \gg 0$. The constraints are twice continuously differentiable. Thus, we can use a calculus based approach to solve the UMP by employing the Kuhn-Tucker conditions. The Lagrangian for the UMP is
\[ L = U(x, \bar{p}, p) + \lambda (w - p \cdot x), \]
where $\lambda \geq 0$ is the Lagrangian multiplier and $U$ is defined in (2.1). The solution to the UMP is the vector of Marshallian demand curves $x(\bar{p}, p, w)$.

The first order Kuhn-Tucker conditions, evaluated at the optimal consumption bundle $x(\bar{p}, p, w)$ are
\[ \frac{\partial L}{\partial x_i} = \frac{\partial u(x(\bar{p}, p, w))}{\partial x_i} + \mu g_i - \lambda p_i \leq 0, i = 1, 2. \tag{3.3} \]
\[ x_i \frac{\partial L}{\partial x_i} = 0, i = 1, 2. \]

Since preferences are locally-nonsatiated, the budget constraint binds, so $p \cdot x = w$.\(^5\)

The second order condition is that the relevant bordered Hessian matrix, $H$, evaluated at the optimal solution, $x$, be negative semi-definite, where
\[ H = \begin{bmatrix} 0 & D_x B_{p,w} \\ D_x B_{p,w} & D^2 L(x, \lambda) \end{bmatrix} \]

Since
\[ \frac{\partial^2 L}{\partial x_i \partial x_j} = \frac{\partial^2 u(x(\bar{p}, p, w))}{\partial x_i \partial x_j}, \]
concavity of $u$ is sufficient to ensure that the second order conditions are satisfied for the extended preferences $\succeq_{\bar{p},p}$.

At an interior solution, from the first order conditions, we get
\[ \frac{\partial u(x(\bar{p}, p, w))}{\partial x_i} + \mu g_i = \frac{p_i}{p_j}, \tag{3.4} \]

\(^5\)For a formal proof, see Proposition 2 below.
The utility function in the 2 possible cases is given by:

Let $p$ be bounded above by $U(x, y)$. The second line follows from the assumed concavity of $U$.

**Proposition 1**  
(a) Let $X$ be a convex set of alternatives. If $u : X \rightarrow \mathbb{R}$ is concave then $U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$, is a concave function of alternatives in $X$.  
(b) A solution always exists for any price vector $p \gg 0$.  
(c) The set of solutions to the UMP is a convex set. If $U$ is strictly quasiconcave, the solution is unique.

Proof: (a) By definition, $U(x, \bar{p}, p) = u(x_1, x_2) + \mu [g_1(\bar{p}, p)x_1 + g_2(\bar{p}, p)x_2]$. Let $x, y \in X$ and let $x^c = \alpha x + (1 - \alpha)y$, $\alpha \in [0, 1]$.

$$U(x^c, \bar{p}, p) = u(x^c) + \mu [g_1(\bar{p}, p)x^c_1 + g_2(\bar{p}, p)x^c_2]$$

$$\geq \alpha u(x) + (1 - \alpha)u(y) + \mu [g_1(\bar{p}, p)(\alpha x_1 + (1 - \alpha)y_1) + g_2(\bar{p}, p)(\alpha x_2 + (1 - \alpha)y_2)]$$

$$= \alpha (u(x) + \mu [g_1(\bar{p}, p)x_1 + g_2(\bar{p}, p)x_2]) + (1 - \alpha) (u(y) + \mu [g_1(\bar{p}, p)y_1 + g_2(\bar{p}, p)y_2])$$

$$= \alpha U(x, \bar{p}, p) + (1 - \alpha)U(y, \bar{p}, p).$$

The second line follows from the assumed concavity of $u$. It follows that $U(x^c, \bar{p}, p) \geq \alpha U(x, \bar{p}, p) + (1 - \alpha)U(y, \bar{p}, p)$, so $U$ is concave.

(b) $U$ defined in (2.1) is a continuous function. If $p \gg 0$ then $B_{p,w}$ is compact; $B_{p,w}$ is bounded above by $\left(\frac{w}{p_1}, \frac{w}{p_2}\right)$ and it is a closed set. A continuous function reaches a maximum on a compact set, hence, a solution exists.

(c) Suppose that $S(p, w) = \{x \in X : x \text{ solves UMP}\}$ is the set of solutions. Let $x, y \in S(p, w)$. By definition, $U(x) = U(y) = U^*$. Now consider $x^c = \alpha x + (1 - \alpha)y$, $\alpha \in [0, 1]$. By quasiconcavity of $U$, $U(x^c) \geq U^*$. Furthermore $p.x^c = \alpha p.x + (1 - \alpha)p.y \leq w$, thus, $x^c$ is affordable. Hence, $x^c \in S(p, w)$. If $U$ is strictly convex, then $U(x^c) > U^*$, which violates the assumption that $x, y \in S(p, w)$; in this case the maximum element is a singleton. ■

3.1. A Special Case: $p_2 = \bar{p}_2$

Let $p_2 = \bar{p}_2$ but the price of good 1, $p_1$, is allowed to differ from the reference price, $\bar{p}_1$. The utility function in the 2 possible cases is given by:

$$\begin{cases} 
U(x, \bar{p}, p) = u(x_1, x_2) + \mu (\bar{p}_1 - p_1)x_1 & \text{if } \bar{p}_1 \geq p_1 \ (\text{Domain of Gains}) \\
U(x, \bar{p}, p) = u(x_1, x_2) - \mu \theta (p_1 - \bar{p}_1)x_1 & \text{if } \bar{p}_1 < p_1 \ (\text{Domain of Losses})
\end{cases}.$$
The MRS in each of the two cases is given by:

\[
MRS_{12} = \begin{cases} 
-\frac{u_1 + \theta(p_1 - p_1)}{u_2} & \text{if } \bar{p}_1 \geq p_1 \\
-\frac{u_1 - \theta(p_1 - p_1)}{u_2} & \text{if } \bar{p}_1 < p_1
\end{cases}.
\]

(3.5)

Figure 3.1 shows two intersecting families of indifference curves corresponding to the following two cases while we hold fixed \(p_2 = \bar{p}_2\).

(1) \(\bar{p}_1 > p_1\): The consumer is in the domain of gains, and the indifference curves are relatively steeper because the marginal utility of \(x_1\) in gains is higher.

(2) \(\bar{p}_1 < p_1\): The consumer is in the domain of losses, and the indifference curves are relatively flatter because the marginal utility of \(x_1\) in losses is lower.

Figures 3.2 and 3.3 show the effects of a decrease and an increase in the price of good 1, respectively, relative to the reference price of good 1. In Figure 3.2, the original price vector, \(p = (\bar{p}_1, \bar{p}_2)\), is also the reference price vector. The budget constraint is shown as the set \(B_{\bar{p},w}\). Since the actual prices and the reference prices coincide, there is no gain-loss utility (we have \(U = u\)) and the corresponding preferences are given by \(\succeq_{p,\bar{p}}\). The utility maximizing consumption bundle is shown by point \(A\) and the corresponding utility maximizing indifference curve is labelled by \(U_{p,\bar{p}}\).

Now consider a fall in the price to \(p = (p_1, \bar{p}_2)\) and \(p_1 < \bar{p}_1\). The new budget constraint is shown by the set \(B_{p,w}\). Since the consumer is now in the domain of gains with respect to
good 1, the marginal rate of substitution changes from $-\frac{u_1}{u_2}$ to $-\frac{u_1 + \mu(p_1 - p_1)}{u_2}$; the marginal utility of good 1 has increased, so the indifference curves become relatively steeper. The new equilibrium is shown by the point $B$; the corresponding indifference curve, is labelled with the utility level $\tilde{U}(\geq \bar{p}, \bar{p})$.

By way of comparison, at the new price vector, $p$, we have also shown the equilibrium in the absence of gain-loss utility at point $C$. The corresponding indifference curve, which is flatter because there is no utility from gains in this case, is labelled by $\tilde{U}(\geq) < \tilde{U}(\geq \bar{p}, \bar{p})$; the two indifference curves intersect at point $D$ on account of context/frame dependence of preferences. Equilibrium $C$ involves a relatively lower consumption of $x_1$ as compared to the consumption bundle at $B$ because it does not take account of the extra utility utility from the price of good 1 being in the domain of gains. Hence, the marginal utility of good 1 and the consumption of good 1 is relatively higher at point $B$. However, this result need to not hold if we also allowed for gain-loss utility for good 2 by allowing for the case $p_2 \neq \bar{p}_2$. In this case the relative gains and losses become important; see Proposition 4 below.

Figure 3.3 shows the case of a rise in price from the actual and reference price $\bar{p} = (\bar{p}_1, \bar{p}_2)$ to the new price $p = (p_1, \bar{p}_2)$, $p_1 > \bar{p}_1$. In the spirit of comparative statics in classical consumer theory, the reference price does not change perhaps because it take time to evolve, while we have a static case. In this case, the original equilibrium consumption bundle at

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6The dynamics of reference price adjustment is still a relatively underexplored area in which there is
the price vector $\overline{p}$ is shown by point $A$; since actual and reference prices equal so there is no gain-loss utility. The equilibrium consumption bundle at the new price vector $p$, when the consumer is in the domain of losses with respect to good 1, is shown at the point $B$; the corresponding indifference curve $\overline{U} (\succ_{\overline{p},p})$ is relatively flatter because it reflects a lower marginal utility of good 1. By way of comparison we have drawn another indifference curve in the absence of gain-loss utility, which reflects classical preferences $\succeq$; this is tangent to the new budget line $B_{p,w}$ and attains a utility level $\overline{U} (\succ)$. Since these preferences do not take account of the loss in the domain of prices, we have $\overline{U} (\succeq) > U (\succ_{\overline{p},p})$; the two indifference curves intersect at $D$.

**Example 6** (Substitute goods and reference dependence): Suppose that $u (x) = ax_1 + bx_2$, $a = 2, b = 1$. Consider first, classical preferences $\succeq$ when there is no gain-loss utility. The absolute value of the marginal rate of substitution between goods 1 and 2 is $|MRS_{12}| = \frac{a}{b} = 2$. Let the actual and reference prices of good 2 be equal and normalized to 1, i.e., $p_2 = \overline{p}_2 = 1$. Let the initial price of good 1 equal its reference price, $\overline{p}_1 = p_1 = 4$, so $\frac{\overline{p}_1}{p_2} = \frac{\overline{p}_1}{\overline{p}_2} = 4$. We have $\frac{\overline{p}_1}{p_2} > |MRS_{12}|$ (the indifference curves are flat relative to the budget constraint), hence, the optimal solution is given by $(x_1, x_2) = (0, w)$ and shown at point $A$ in the Figure. Suppose now that the price of good 1 falls to 3 but the price of good still not a well-agreed upon set of stylized facts that theorists can use in their work.
2 does not change, so the new price ratio is \( \frac{p_1}{p_2} = 3 \). In the absence of gain-loss utility \( \frac{p_1}{p_2} > |MRS_{12}| \), so the optimum continues to be at point A.

Introduce now, gain-loss utility but the reference prices are assumed to stay unchanged. There is no gain-loss utility at the original prices \( p_1, p_2 \) because the actual and reference prices are identical. But the reduction in \( p_1 \) to \( p_1 = 3 \) pushes the individual into the domain of gains with respect to \( x_1 \). At the new price ratio, \( \frac{p_1}{p_2} = 3 \), we have \( |MRS_{12}| = \frac{a+\mu(4-3)}{b} = 2 + \mu \). If \( \mu = 1 \), then we have \( \frac{p_1}{p_2} = |MRS_{12}| \) and there is a continuum of solutions; any mix of expenditures on the two goods that satisfies the budget constraint is optimal. If \( \mu > 1 \), say \( \mu = 2 \), which is the case shown in Figure 3.4, then \( \frac{p_1}{p_2} < |MRS_{12}| \) and the unique optimum flips to the other corner, \((x_1, x_2) = (w, 0)\) shown by the point B.

Hence, a consideration of gain-loss utility alters the equilibrium outcomes from one corner solution to the other, but there is no change in the solution in the absence of gain-loss utility.

### 3.2. Properties of the Marshallian demand curves

**Proposition 2**: Suppose that \( U \) is a continuous utility function that represents the locally non-satiated preferences \( \succeq_{\bar{p},p} \) on the consumption set \( X \). The Marshallian demand curves \( x(\bar{p},p,w) \) have the following properties.

(i) \( x(\bar{p},p,w) \) is not homogenous of degree zero in \( p,w \).

(ii) Walras’ law holds, i.e., \( p.x = w \).

(iii) \( x(\bar{p},p,w) \) is continuous in \( p \).
Proof: Consider an initial vector \((\overline{p}, w)\) where prices equal the reference prices. Now consider a new vector \((tp, tw)\), where \(t > 0\) is a scalar and the vector of new prices is \(p = t\overline{p}\) but the reference price vector continues to be \(\overline{p}\). By definition, the budget set \(B_{\overline{p}, w} = \{x \in X : \overline{p}x \leq w\}\) is identical to \(B_{p, tw} = \{x \in X : t\overline{p}x \leq tw\}\). Hence, the constraint set does not change. The marginal rate of substitution between the two goods at the price vector \(tp\) is \(\text{MRS}_{12}(x, \overline{p}, tp) = -\frac{u_1(x) + \mu_1(\overline{p}, tp)}{u_2(x) + \mu_2(\overline{p}, tp)}\), which varies with \(t\). Hence, the optimal consumption bundle is not invariant with respect to \(t\).

(ii) Suppose that Walras’ law does not hold and at the optimal bundle, \(x^*\), we have \(p.x^* < w\). Then local non-satiation implies that for any \(\varepsilon > 0\) we can find another bundle \(y\) such that \(\|x^* - y\| < \varepsilon\), and \(y >_{\overline{p}, p} x^*\). Furthermore, since \(x^*\) is an interior point of a closed set, the bundle \(y\) is affordable, i.e., \(p.y \leq w\). But this contradicts the original assumption that \(x^*\) is optimal.

(iii) In the UMP we maximize a continuous function on a compact interval. Hence, from the theorem of the maximum it follows that \(x(\overline{p}, p, w)\) is continuous in \(p\).

3.3. Indirect utility function and its properties

Define the indirect utility in the presence of gain-loss preferences \(\succeq_{\overline{p}, p}\) by

\[
V(\overline{p}, p, w) = u(x(\overline{p}, p, w)) + \mu [g_1(\overline{p}, p)x_1(\overline{p}, p, w) + g_2(\overline{p}, p)x_2(\overline{p}, p, w)].
\] (3.6)

**Proposition 3**: Suppose that \(U\) is a continuous utility function that represents the locally non-satiated preferences \(\succeq_{\overline{p}, p}\) on the consumption set \(X\). The indirect utility function \(V(\overline{p}, p, w)\) corresponding to \(U\) has the following properties.

(i) \(V(\overline{p}, p, w)\) is not homogeneous of degree zero in \((p, w)\).

(ii) \(V\) is (a) increasing in \(w\) and (b) decreasing in \(p\).

(iii) \(V\) is quasiconvex in \(p\).

(iv) \(V\) is continuous in \(p\).

Proof: (i) The Marshallian demand curves \(x_1(\overline{p}, p, w)\), \(x_2(\overline{p}, p, w)\) are not homogeneous of degree zero in \((p, w)\); see Proposition 2. Hence, \(V(\overline{p}, tp, tw) \neq V(\overline{p}, p, w)\).

(ii) Suppose that \(x(\overline{p}, p, w)\) is the optimal consumption bundle at \((\overline{p}, p, w)\).

(a) It follows that \(x(\overline{p}, p, w)\) must be affordable: \(p.x(\overline{p}, p, w) \leq w\). Now suppose that \(w\) increases to \(w'\). Then \(p.x(\overline{p}, p, w) < w'\). Due to local non-satiation, for every \(\varepsilon > 0\), there exists \(y \in X\), \(\|y - x\| < \varepsilon\), such that \(y >_{\overline{p}, p} x\). Since \(V\) represents the underlying preferences, \(V(\overline{p}, p, w) < V(\overline{p}, p, w')\).

(b) Since Walras’ Law holds, so by definition \(p.x(\overline{p}, p, w) = w\). Now consider a new price vector, \(p'\), in which at least one price is lower, so \(p \gg p'\). Thus, \(p'.x(\overline{p}, p, w) < p.x(\overline{p}, p, w) = w\). Since \(p'.x(\overline{p}, p, w) < w\), local non-satiation ensures that for every \(\varepsilon > 0\),
there exists \( y \in X, \|y - x(\bar{p}, p, w)\| < \varepsilon \), such that \( y \succ x(\bar{p}, p, w) \). Thus, \( V(\bar{p}, p', w) > V(\bar{p}, p, w) \), so \( V \) is non-increasing in \( p_1 \).

(iii) We shall exploit the property that the lower contour sets of quasiconvex functions are convex. Consider prices \( p, p' \) and the reference price is fixed at \( \bar{p} \). The optimal bundles at \((\bar{p}, p, w)\) and \((\bar{p}, p', w)\) are \( x = x(\bar{p}, p, w) \) and \( x' = x'(\bar{p}, p', w) \), respectively, and the respective indirect utilities are \( V = V(\bar{p}, p, w) \) and \( V' = V'(\bar{p}, p', w) \). Suppose that \( V \leq \bar{V} \) and \( V' \leq \bar{V} \) and \( p^c = \alpha p + (1 - \alpha)p', \alpha \in [0, 1] \). We need to show that \( V^c = V^c(\bar{p}, p^c, w) \leq \bar{V} \). Since income stays fixed at \( w \), any bundle \( \hat{x} \) that is affordable at \((\bar{p}, p^c, w)\) must satisfy \( p^c.\hat{x} \leq w \), or \( (\alpha p + (1 - \alpha)p').\hat{x} \leq w \), which can be written as

\[
\alpha p.\hat{x} + (1 - \alpha)p'.\hat{x} \leq w.
\]

This implies that \( p.\hat{x} \leq w \) or \( p'.\hat{x} \leq w \) or both. If \( p.\hat{x} \leq w \) then \( \hat{x} \) was affordable at the price \( p \) so \( U(\hat{x}, \bar{p}, p) \leq U(x, \bar{p}, p) = V \leq \bar{V} \). Since \( \hat{x} \) was arbitrary, this holds true when \( \hat{x} \) is also the optimal bundle at \((\bar{p}, p^c, w)\). Thus, \( V^c \leq \bar{V} \). Analogously, if \( p'.\hat{x} \leq w \) then we can show that \( V^c \leq \bar{V} \).

(iv) The UMP has the maximization of a continuous function on a compact interval. Hence, from the theorem of the maximum it follows that \( V(\bar{p}, p, w) \) is continuous in \( p \).
3.4. Kinked demand curves

Suppose that the demand curves are differentiable under the classical preferences, \(\succeq\).\(^7\) At an interior solution, the first order condition (3.3) gives \(\frac{\partial L}{\partial x_i} = \frac{\partial u(x(\bar{p}, p, w))}{\partial x_i} + \lambda g_i = 0\). Implicitly differentiate to get

\[
\frac{\partial x(p, p, w)}{\partial p_i} = \left( - \frac{\partial^2 u(x(\bar{p}, p, w))}{\partial x_i^2} \right)^{-1} \left( \frac{\partial^2 u(x(\bar{p}, p, w))}{\partial p_i \partial x_i} - \frac{\partial \lambda p_i}{\partial p_i} + \frac{\partial g_i}{\partial p_i} \right),
\]

where the function \(u\) is twice continuously differentiable. Suppose that \(p = (p_i, p_j), i \neq j\). Hence, it suffices to check if the function \(\frac{\partial g_i}{\partial p_i}\) is continuous at \(\bar{p}_i\). Let \(\{p_i^-\}\) be a sequence that converges from the left to \(\bar{p}_i\) and let \(\{p_i^+\}\) be a sequence that converges from the right to \(\bar{p}_i\). Thus,

\[
\lim_{p_i^- \to \bar{p}_i} \frac{\partial g_i}{\partial p_i} = -\mu \neq \lim_{p_i^+ \to \bar{p}_i} \frac{\partial g_i}{\partial p_i} = -\theta \mu.
\]

Figure 3.5 shows a kink in the Marshallian demand curve at the reference price, \(\bar{p}_1\); from Proposition 2 the demand curves are continuous. Since \(\theta > 1\) (and \(p_1\) is measured on the vertical axis) the portion of the demand curve above the price \(\bar{p}_1\) is relatively flatter.

3.5. Money illusion

In the classical theory of the consumer, there is no gain-loss utility and so consumers do not suffer from any money illusion. However, empirical evidence supports the presence of money illusion (Shafir et al., 1997). We now show that consumers suffer from money illusion in the presence of gain-loss utility.

Suppose that the vector \((p, w)\) changes to \((tp, tw)\), \(t > 0\). There is no change in the budget set because \(B_{p,w} = B_{tp, tw}\). However, the marginal rate of substitution between the two goods is given by

\[
MRS_{12}(x, \bar{p}, p) = -\frac{u_1(x)}{u_2(x)} + \frac{\mu g_1(\bar{p}_1, p_1)}{\mu g_2(\bar{p}_2, p_2)}.
\]

(3.7)

Unless \(\bar{p} = p\) (so \(g_i = 0\) for \(i = 1, 2\)) there is no presumption that \(MRS_{12}(x, \bar{p}, p) = MRS_{12}(x, p, tp)\). Hence, there will be some reallocation of the optimal consumption bundle. This is a form of money illusion. This is by no means the only framework that can produce money illusion. For instance, money illusion can also arise in the presence of limited attention (Gabaix, 2014).

**Example 7**: Suppose that \(\bar{p}_i \geq p_i, i = 1, 2\). Then,

\[
MRS_{12}(x, p, p) \geq MRS_{12}(x, \bar{p}, tp) \Leftrightarrow \mu [p_2 u_1 - p_1 u_2] + \mu^2 [p_2 \bar{p}_1 - p_1 \bar{p}_2] \geq 0.
\]

\(^7\)Monteiro et al. (1996) show that the following conditions are sufficient for differentiability of demand:

(i) Utility is separable.  
(ii) Utility is concave and quasilinear.  
(iii) Utility is differentiable and marginal utilities are pointwise Lipschitzian.
In the absence of gain-loss utility, $\mu = 0$, so $\text{MRS}_{12}(x, \bar{p}, p) = \text{MRS}_{12}(x, \bar{p}, tp)$, hence, there is no money illusion.

### 3.6. An Example: Quasilinear Preferences

Suppose that classical preferences are quasilinear:

$$u(x) = x_1 + \ln x_2. \quad (3.8)$$

These preferences satisfy the sufficient conditions in Monteiro et al. (1996) for the differentiability of demand functions under classical preferences. From (3.3), the first order condition for an interior solution is

$$\begin{align*}
1 + \mu g_1 - \lambda p_1 &= 0 \Rightarrow \lambda = \frac{(1+\mu g_1)}{p_1} \\
\frac{1}{x_2} + \mu g_2 - \lambda p_2 &= 0 \Rightarrow \lambda = \frac{x_2 + \mu g_2}{p_2}.
\end{align*} \quad (3.9)$$

Eliminating $\lambda$ we get that $\frac{(1+\mu g_1)}{p_1} = \frac{1}{x_2} + \mu g_2$. Solving out for the Marshallian demand for good 2, we get

$$x_2(\bar{p}, p, w) = \frac{p_1}{p_2 (1 + \mu g_1) - \mu p_1 g_2}, \quad (3.10)$$

which is independent of $w$ because of quasilinear preferences. From Proposition 2(ii) Walras’ Law holds. Hence, the budget constraint binds when evaluated at the optimal solution $x_1 = x_1(\bar{p}, p, w)$, $x_2 = x_2(\bar{p}, p, w)$, so

$$p_1 x_1(\bar{p}, p, w) + p_2 x_2(\bar{p}, p, w) = w. \quad (3.11)$$

Substituting $x_2$ from (3.10) into (3.11) and solving out for $x_1$ we get

$$x_1(\bar{p}, p, w) = \frac{w}{p_1} - \frac{p_2}{p_2 (1 + \mu g_1) - \mu p_1 g_2}. \quad (3.12)$$

Substituting (3.10) and (3.12) in (3.8) we get the indirect utility function

$$V(\bar{p}, p, w) = \frac{w}{p_1} - \frac{p_2}{p_2 (1 + \mu g_1) - \mu p_1 g_2} + \ln \frac{p_1}{p_2 (1 + \mu g_1) - \mu p_1 g_2}. \quad (3.13)$$

A tractable special case arises when $p_2 = \bar{p}_2$ so that $g_2(\bar{p}_2, \bar{p}_2) = 0$. We summarize the solution in this case:

$$\begin{align*}
x_1(\bar{p}, p, w) &= \frac{w}{p_1} - \frac{1}{1 + \mu g_1} \\
x_2(\bar{p}, p, w) &= \frac{p_1}{p_2 (1 + \mu g_1)} \\
V(\bar{p}, p, w) &= \frac{w}{p_1} - \frac{1}{1 + \mu g_1} + \ln \frac{p_1}{p_2 (1 + \mu g_1)}.
\end{align*} \quad (3.14)$$
The Marshallian demand curves and the indirect function are continuous in $p_1$ but there is a kink in the Marshallian demand curves for both goods at a price $p_1 = \bar{p}_1$. To see this, first write the derivatives of the Marshallian demands with respect to $p_1$.

\[
\begin{align*}
\frac{\partial x_1}{\partial p_1} &= -\frac{w}{p_1^2} + \frac{1}{(1+\mu p_1)} \mu \frac{\partial g_1}{\partial p_1} , \\
\frac{\partial x_2}{\partial p_1} &= \frac{1}{p_2(1+\mu p_1)^2} \left( 1 + \mu g_1 - p_1 \mu \frac{\partial g_1}{\partial p_1} \right) .
\end{align*}
\]

Let $\{p_{1n}^-\}$ be a sequence of prices of good 1 that converges from the left to $p_1$ and let $\{p_{1n}^+\}$ be a price sequence that converges from the right to $\bar{p}_1$. Using (2.3), we get

\[
\lim_{p_{1n}^- \to p_1} \frac{\partial x_1}{\partial p_1} = -\frac{w}{p_1^2} - \mu \neq \lim_{p_{1n}^+ \to p_1} \frac{\partial x_1}{\partial p_1} = -\frac{w}{p_1^2} - \theta \mu .
\]

\[
\lim_{p_{1n}^- \to p_1} \frac{\partial x_2}{\partial p_1} = \frac{1}{p_2^2} (1 + \mu p_1) \neq \lim_{p_{1n}^+ \to p_1} \frac{\partial h_2}{\partial p_1} = \frac{1}{p_2} (1 + \theta p_1 \mu) .
\]

Hence, there is a kink in the Marshallian demand curves for both goods at a price $p_1 = \bar{p}_1$; see Figure 3.5.

Money illusion can be easily verified from (3.10) and (3.12). It is sufficient to note that $f(p_1, p_2) = p_2 g_1(p_1, p_1) - p_1 g_2(p_2, p_2) \neq t f(p_1, p_2)$, $t > 0$.

Relative to classical preferences, there are two extra parameters when we introduce gain-loss utility, $\mu$ and $\theta$. The next proposition considers the comparative static effects of these parameters on the Marshallian demands for quasilinear preferences. When gain-loss effects are non-zero for both goods then the effects of $\mu$ and $\theta$ on the Marshallian demands depend on the relative strengths of these effects on the two goods.

**Proposition 4**: Suppose that preferences are quasilinear and are given by (3.8).

(i) Suppose that both prices are in the domain of gains, i.e., $\bar{p}_1 \geq p_1, p_2 \geq p_2$.

If $p_2 \bar{p}_1 - p_1 \bar{p}_2 > 0$ \quad then \quad \begin{align*}
\frac{\partial x_1}{\partial \theta} &> 0 , \quad \frac{\partial x_2}{\partial \theta} < 0 \\
\frac{\partial x_1}{\partial \mu} &< 0 , \quad \frac{\partial x_2}{\partial \mu} > 0
\end{align*}

In either case, $\frac{\partial x_1}{\partial \theta} = \frac{\partial x_2}{\partial \theta} = 0$.

(ii) Suppose that the price of good 1 is in the domain of losses and the price of good 2 is in the domain of gains, i.e., $p_1 > \bar{p}_1, p_2 \geq p_2$, then

$\frac{\partial x_1}{\partial \theta} < 0 , \quad \frac{\partial x_2}{\partial \theta} > 0$,

and

if $\theta p_2 \bar{p}_1 - p_1 \bar{p}_2 - (\theta - 1) p_1 p_2 > 0$ \quad then \quad \begin{align*}
\frac{\partial x_1}{\partial \theta} &> 0 , \quad \frac{\partial x_2}{\partial \theta} < 0 \\
\frac{\partial x_1}{\partial \mu} &< 0 , \quad \frac{\partial x_2}{\partial \mu} > 0
\end{align*}
(iii) Suppose that the price of good 1 is in the domain of gains and the price of good 2 is in the domain of losses, i.e., $p_1 > \bar{p}_1$, $p_2 > \bar{p}_2$, then

$$\frac{\partial x_1}{\partial \theta} > 0, \quad \frac{\partial x_2}{\partial \theta} < 0,$$

and

$$\text{if } p_2 \bar{p}_1 - \theta p_1 \bar{p}_2 + (\theta - 1) p_1 p_2 \begin{cases} > 0 & \text{then} \quad \frac{\partial x_1}{\partial \mu} > 0, \quad \frac{\partial x_2}{\partial \mu} < 0 \\ < 0 & \text{then} \quad \frac{\partial x_1}{\partial \mu} < 0, \quad \frac{\partial x_2}{\partial \mu} > 0 \end{cases}$$

(iv) Suppose that the price of both goods is in the domain of losses, i.e., $p_1 > \bar{p}_1$, $p_2 > \bar{p}_2$.

$$\text{If } p_2 \bar{p}_1 - p_1 \bar{p}_2 \begin{cases} > 0 \quad & \text{then} \quad \frac{\partial x_1}{\partial \mu} > 0, \quad \frac{\partial x_2}{\partial \mu} < 0, \quad \frac{\partial x_1}{\partial \theta} > 0, \quad \frac{\partial x_2}{\partial \theta} < 0 \\ < 0 \quad & \text{then} \quad \frac{\partial x_1}{\partial \mu} < 0, \quad \frac{\partial x_2}{\partial \mu} > 0, \quad \frac{\partial x_1}{\partial \theta} < 0, \quad \frac{\partial x_2}{\partial \theta} > 0 \end{cases}.$$ 

Proof: Let $\phi(\bar{p}, p, \mu, \theta) = \mu(p_2 g_1 - p_1 g_2)$. Then we can write the Marshallian demands as

$$\begin{cases} x_1(\bar{p}, p, w) = \frac{w}{p_1} - \frac{p_2}{p_1 + p_2 + \phi(\bar{p}, p, \mu, \theta)} \\ x_2(\bar{p}, p, w) = \frac{\mu}{p_1} \frac{p_2}{p_1 + p_2 + \phi(\bar{p}, p, \mu, \theta)} \end{cases}$$

(3.15)

Thus,

$$\frac{\partial x_1}{\partial \phi} > 0, \quad \frac{\partial x_2}{\partial \phi} < 0.$$ 

(3.16)

There are four possible states of the world and the value of $\phi(\bar{p}, p, \mu, \theta)$ in each state is given as follows.

$$\phi(\bar{p}, p, \mu, \theta) = \begin{cases} \mu(p_2 \bar{p}_1 - p_1 \bar{p}_2) & \bar{p}_1 \geq p_1, \quad \bar{p}_2 \geq p_2 \\ \mu(-\theta p_2 (p_1 - \bar{p}_1) - p_1 (\bar{p}_2 - p_2)) & p_1 > \bar{p}_1, \quad \bar{p}_2 > p_2 \\ \mu(p_2 (\bar{p}_1 - p_1) + \theta p_1 (p_2 - \bar{p}_2)) & \bar{p}_1 \geq p_1, \quad p_2 > \bar{p}_2 \\ \mu \theta (p_2 \bar{p}_1 - p_1 \bar{p}_2) & p_1 > \bar{p}_1, \quad p_2 > \bar{p}_2 \end{cases}$$

(3.17)

From (3.15), (3.16), (3.17) we get the required result. 

Consider the condition $p_2 \bar{p}_1 - p_1 \bar{p}_2 \geq 0$ in Proposition 4(i). Both prices are in gains. Yet depending on the parameter values, we get $\frac{\partial x_i}{\partial \mu} \geq 0, \quad i = 1, 2$. For $\frac{\partial x_1}{\partial \mu} < 0$ we need $p_2 \bar{p}_1 - p_1 \bar{p}_2 > 0$. This condition requires that $\bar{p}_1$ be high and $p_1$ low (so the gain $\bar{p}_1 - p_1$ is relatively high) and that $p_2$ be high but $\bar{p}_2$ be low (so the gain on the second good, $p_2 - \bar{p}_2$, is relatively low). Thus, the relative gains influence the marginal effects on the demand for the two goods. This reasoning may be used to from the relevant intuition about all the comparative statics with respect to $\mu$. When only the price of good 1 is in the domain of losses, as in Proposition 4(ii), then $\frac{\partial x_1}{\partial \theta} < 0$. However, when the prices of both goods are in losses, as in Proposition 4(iv), then $\frac{\partial x_1}{\partial \theta} \geq 0$ depending on the relative losses on the two goods. As in our discussion of Proposition 4(i) above, this depends on $p_2 \bar{p}_1 - p_1 \bar{p}_2 \geq 0$. 

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From (3.15), (3.17) we get

\[
x_1(\bar{p}, p, w) = \begin{cases} 
\frac{w}{p_1} - \frac{1}{1+\mu(p_1 - p_2^2p_2)} & p_1 \geq p_1, \bar{p}_2 \geq p_2 \\
\frac{w}{p_1} - \frac{1}{1+\mu(-\theta(p_1-p_1)-p_1^2)} & p_1 > \bar{p}_1, \bar{p}_2 \geq p_2 \\
\frac{w}{p_1} - \frac{1}{1+\mu(-\theta(p_1-p_1)+\theta^2)} & p_1 \geq p_1, p_2 > \bar{p}_2 \\
\frac{w}{p_1} - \frac{1}{1+\mu(\theta p_2^2-p_2^2)} & p_1 > \bar{p}_1, p_2 > \bar{p}_2 
\end{cases}
\]

Differentiating with respect to \(p_1\) we get that \(\frac{\partial x_1}{\partial p_1} < 0\) except possibly for the third case where good 1 is in gains but good 2 is in a loss: \(\bar{p}_1 \geq p_1, p_2 > \bar{p}_2\). In this case, the condition \(\theta (1 - \frac{p_2}{p_1}) - 1 < 0\) is sufficient for \(\frac{\partial x_1}{\partial p_1} < 0\).

From (3.15), (3.17) we get

\[
x_2(\bar{p}, p, w) = \begin{cases} 
\frac{p_1}{p_2+p_2(p_2^2p_2-p_1)} & \bar{p}_1 \geq p_1, \bar{p}_2 \geq p_2 \\
\frac{p_1}{p_2+p_2(-\theta p_2(p_1-p_1)-p_1(p_2-p_2))} & p_1 > \bar{p}_1, \bar{p}_2 \geq p_2 \\
\frac{p_1}{p_2+p_2(-\theta p_2(p_1-p_1)+\theta p_1(p_2-p_2))} & p_1 \geq p_1, p_2 > \bar{p}_2 \\
\frac{p_1}{p_2+p_2(\theta p_2-p_2)} & p_1 > \bar{p}_1, p_2 > \bar{p}_2 
\end{cases}
\]

Differentiating with respect to \(p_2\) we get that \(\frac{\partial x_2}{\partial p_2} < 0\), except possibly for the second case where good 1 is in losses but good 2 is in a gains: \(p_1 > \bar{p}_1, p_2 \geq p_2\). In this case, the condition \(\theta (1 - \frac{p_1}{p_1}) - 1 < 0\) is sufficient for \(\frac{\partial x_2}{\partial p_2} < 0\). Thus, for quasilinear preferences the Marshallian demand curves are downward sloping in most cases. However, the possibility \(\frac{\partial x_1}{\partial p_1} > 0\) and \(\frac{\partial x_2}{\partial p_2} > 0\) arises mainly from the relative domains of prices of the two goods, rather than from classical considerations that underlie the Giffen good case.

**Remark 1**: We cannot compare the utility levels under the indirect utility functions under gain-loss utility, \(V(\bar{p}, p, w)\), and in the absence of gain-loss utility, \(\tilde{V}(p, w)\), in order to compare consumer welfare. The reason is that these two utility functions represent different preferences, \(\succeq_{\bar{p}, p}\) and \(\succeq_{p}\). We have

\[
\begin{cases} 
\succeq: & \tilde{V}(p, w) = \frac{w}{p_1} - 1 + \ln \frac{p_1}{p_2} \\
\succeq_{\bar{p}, p}: & V(\bar{p}, p, w) = \frac{w}{p_1} - \frac{1}{1+\mu g_1} + \ln \frac{p_1}{p_2(1+\mu g_1)} 
\end{cases}
\]

One may expect that in the domain of gains we would have \(V(\bar{p}, p, w) > \tilde{V}(p, w)\) and in the domain of losses, \(V(\bar{p}, p, w) < \tilde{V}(p, w)\). A simple calculation shows that \(V(\bar{p}, p, w) - \tilde{V}(p, w) = \frac{1}{1+\mu} - \ln(1+x)\) where \(x = \mu g_1\). One can check that \(V(\bar{p}, p, w) < \tilde{V}(p, w)\) whenever the consumer is strictly in the domain of gains or losses.

### 3.7. Marshallian demands with and without gain-loss utility

Let \(w = e(\bar{p}, p, U)\). Denote the Marshallian demand for good 1 under no gain-loss utility (\(\mu = 0\)) by \(\bar{x}_1(p, U)\) and under gain-loss utility by \(x_1^*(\bar{p}, p, w)\). Using the equilibrium
condition \( MRS_{12} = -\frac{p_1}{p_2} \) we can compare the levels of Marshallian demands in the presence and in the absence of gain-loss utility. In the absence of gain-loss utility, we have \( \frac{u_1(x)}{u_2(x)} = \frac{p_1}{p_2} \) evaluated at the optimal consumption bundle \( \bar{x} \). Keeping fixed the consumption bundle at \( \bar{x} \), in the presence of gain-loss utility we wish to check if
\[
\frac{u_1(\bar{x}) + \mu g_1}{u_2(\bar{x}) + \mu g_2} \geq \frac{u_1(\bar{x})}{u_2(\bar{x})} = \frac{p_1}{p_2}.
\]
(3.18)

A simple calculation shows that
\[
\frac{u_1(\bar{x}) + \mu g_1}{u_2(\bar{x}) + \mu g_2} \geq \frac{u_1(\bar{x})}{u_2(\bar{x})} \iff u_2(\bar{x}) g_1 \geq u_1(\bar{x}) g_2.
\]
(3.19)

Suppose that the actual and reference price of good 2 is equal \( (\bar{p}_2 = p_2) \), then \( g_2 = 0 \). In this case
\[
\begin{align*}
\bar{p}_1 > p_1 & \Rightarrow u_2(\bar{x}) g_1 > u_1(\bar{x}) g_2 = 0 \Rightarrow \frac{u_1(\bar{x}) + \mu g_1}{u_2(\bar{x}) + \mu g_2} > \frac{u_1(\bar{x})}{u_2(\bar{x})} \\
\bar{p}_1 < p_1 & \Rightarrow u_2(\bar{x}) g_1 < u_1(\bar{x}) g_2 = 0 \Rightarrow \frac{u_1(\bar{x}) + \mu g_1}{u_2(\bar{x}) + \mu g_2} < \frac{u_1(\bar{x})}{u_2(\bar{x})}
\end{align*}
\]
(3.20)

From (3.18)-(3.20) the following conclusion is obvious when \( \bar{p}_2 = p_2 \); Figure 3.6 illustrates this special case.

I: Domain of (strict) gains \( (\bar{p}_1 > p_1) \): In this case we have \( \frac{u_1(\bar{x}) + \mu (\bar{p}_1 - p_1)}{u_2(\bar{x})} > \frac{p_1}{p_2} \). Equilibrium requires an increase in \( x_1 \) and a decrease in \( x_2 \), which implies that \( \bar{x}_1 (p, w) < x^*_1 (p, p, w) \). Since the budget constraint must balance, we have \( x^*_2 (\bar{p}, p, w) < \bar{x}_2 (p, w) \). This is the case shown in Figure 3.2.
II. Domain of (strict) losses ($\bar{p}_1 < p_1$): In this case we have $\frac{u_1(\bar{x}) - \theta u_1(p_1 - p_1)}{u_2(\bar{x})} < \frac{p_1}{p_2}$. Equilibrium requires a decrease in $x_1$ and an increase in $x_2$, hence, $\bar{x}_1(p, w) > x_1^*(\bar{p}, p, w)$. Since the budget constraint must hold, we get $x_2^*(\bar{p}, p, w) > x_2(p, w)$. This is the case shown in Figure 3.3.

III. Neither gains nor losses ($\bar{p}_1 = p_1$): In this case $u_1(\bar{x}) = u_2(\bar{x}) = p_1 p_2$ in the presence and in the absence of gain-loss utility, while the budget constrain is unaffected, hence, $\bar{x} = x^*$.

**Example 8** (Quasilinear preferences): Consider the example in Section 3.6 when $p_2 = \bar{p}_2$. The solution is summarized in 3.14. The Marshallian demand curves in the presence of gain-loss utility are denoted by $\bar{x}(\bar{p}, p, w)$ and in the absence of gain-loss utility by $\bar{x}(p, w)$.

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$x_1^*(\bar{p}, p, w) = \frac{w}{p_1} - \frac{1}{1 + \mu g_1}$</th>
<th>$\bar{x}_1(p, w) = \frac{w}{p_1} - 1$</th>
<th>$\mu = 0$</th>
<th>$x_2^*(\bar{p}, p, w) = \frac{p_1}{\bar{p}_1(1 + \mu g_1)}$</th>
<th>$\bar{x}_2(p, w) = \frac{p_1}{\bar{p}_1}$</th>
</tr>
</thead>
</table>

When the consumer is in gains $g_1 = p_1 - \bar{p}_1 > 0$, $\frac{1}{1 + \mu g_1} < 1$, so $x_1^*(\bar{p}, p, w) > \bar{x}_1(p, w)$. When the consumer is in losses, $g_1 = -\theta (p_1 - \bar{p}_1) < 0$, $\frac{1}{1 + \mu g_1} > 1$, so $x_1^*(\bar{p}, p, w) < \bar{x}_1(p, w)$.$^8$

The relative shapes of the Marshallian demands shown in Figure 3.6 need not hold when we allow for gain-loss utility in both dimensions. In this case, the outcome is governed by the inequalities in (3.18) and (3.19). The relative strengths of the gain and loss effects for each good will now determine the relative Marshallian demands in the presence and in the absence of gain-loss utility; for the special case of quasilinear preferences, see Proposition 4.

For instance, suppose that $\bar{p}_1 > p_1$ so that we are in the domain of gains for good 1 ($g_1 > 0$). In this case, we may have $\bar{x}_1(p, w) > x_1^*(\bar{p}, p, w)$ (the opposite of the case shown in Figure 3.6) if $g_2 > 0$ but $\bar{p}_1 - p_1$ is small relative to $\bar{p}_2 - p_2$ so that $0 < \frac{u_2(\bar{x})}{u_1(\bar{x})} < \frac{\bar{x}_2}{g_1}$. The intuition is quite simple. The relative gains are higher for good 2, raising its marginal utility relative to good 2, hence, the optimal consumption of good 1 is lower relative to the case where gain-loss utility is absent. Similarly when $\bar{p}_1 < p_1$ but the losses on good 2 are relatively larger (so $p_2 - \bar{p}_2$ is relatively higher) then the consumption of good 1 will be higher relative to the case of no gain-loss utility (this is the opposite of the case shown in Figure 3.6).

### 3.8. Apparent intransitivity of preferences

There have been several demonstrations of inconsistent and intransitive preferences (Lichtenstein and Slovic, 1971; Lichtenstein and Slovic, 2006; Kahneman and Tversky, 2000). Others have argued for maintaining the minimum assumption of consistent preferences,

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$^8$We are assuming that $\frac{1}{1 + \mu g_1} > 0$ otherwise the consumer budget constraint is violated at the optimal consumption bundle.
albeit taking into account the context/frame for a decision into account. This is a recurring theme in the work of Herbert Gintis; see, for instance, Gintis (2009). In this section we show that at least some types of preference inconsistencies may be the result of not taking account of gain-loss utility.

Consider an initial price vector $p^0$ and a final price vector $p^1$; reference prices are captured by the vector $\bar{p}$.

At the initial prices, the family of indifference curves is indexed by $\succeq_{\bar{p},p^0}$. In this case, suppose that the choice data reveals that

$$B \succeq_{\bar{p},p^0} C \succeq_{\bar{p},p^0} A.$$ 

In Figure 3.7, the corresponding family of indifference curves is drawn as the set of three relatively flatter curves. In particular the indifference curve drawn through the consumption bundle $A$ is associated with a level of utility $\tilde{U}(\succeq_{\bar{p},p^0})$.

Now consider a change in the economic environment to $(\bar{p}, p^1, w)$ such that the price of good 1 falls and the price of good 2 does not fall (it may increase). The new family of three indifference curves shown in the diagram is steeper and represents the preferences $\succeq_{\bar{p},p^1}$. A member of this family passing through $A$ is associated with a level of utility $U(\succeq_{\bar{p},p^1})$. Suppose that we now observe the following preference pattern

$$A \succeq_{\bar{p},p^1} C \succeq_{\bar{p},p^1} B.$$ 

Suppose that these two sets of choices are observed by a researcher who does not take account of gain-loss utility and assumes that the underlying preferences are the classical
preferences, \(\succeq\). Then the researcher would conclude from the two set of observations that

\[
\begin{align*}
B &\succeq C \succeq A \\
A &\succeq C \succeq B
\end{align*}
\]  

(3.21)

From the second set of preferences the researcher would conclude that preferences are intransitive because we have the cycle \(B \succeq C \succeq A \succeq B\). Taking account of gain-loss utility, the underlying preferences of the individual are perfectly consistent. We do not claim that this argument can explain all potential preference inconsistencies— but it may explain some.

### 4. Expenditure minimization problem

Suppose that prices are strictly positive, \(p \gg 0\), and we wish to attain a utility level \(U > U(0, \bar{p}, p)\), where \(O = (0, 0)\). The expenditure minimization problem (EMP) finds the minimum level of wealth that is required to attain at least the utility level \(U\). Formally, the EMP involves the choice of a set of Hicksian consumption bundles, \(x^H\), such that

\[
x^H \in \text{arg min}_{x \in X} p.x
\]

subject to

\[
U(x, \bar{p}, p) \geq U.
\]

As in the UMP, \(U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}\) represents continuous, locally non-satiated preferences.

If \(p \gg 0\) and the constraint set is non-empty, then a solution exists under quite general conditions. Note first that \(p.x\) is continuous in \(x\). Let \(\bar{x} \in \mathbb{R}^2\) and let \(U(\bar{x}, \bar{p}, p) \geq U\). Define \(S = \{x : U(x, \bar{p}, p) \geq U \text{ and } p.x \leq p.\bar{x}\}\). (i) \(S\) is non-empty because \(\bar{x} \in S\). (ii) \(S\) is closed. (iii) The set \(S\) is bounded. To see this, let one of the goods \(x_i = 0\), then \(x_j (j \neq i)\) satisfies \(x_j \leq w/p_j\). Thus, \(x_j \leq w/p_j, j = 1, 2\) and given \(p_j > 0, x_j\) is bounded above. Hence, \(S\) is bounded. A continuous function on a closed and bounded set has a minimum value, hence, the EMP has a solution.

**Proposition 5** : Let \(U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}\) represent continuous, locally non-satiated preferences, \(\succeq_{p,p}\) and let \(p \gg 0\).

(i) The constraint in the EMP binds.

(ii) Let \(x^M\) solve the UMP at \((\bar{p}, p, w)\). Then, \(x^M\) also solves the EMP \(\min p.x\) subject to \(U(x, \bar{p}, p) \geq U(x^M, \bar{p}, p)\). Furthermore, \(p.x^M = w\).

(iii) Suppose that \(x^H\) solves the EMP when required utility is \(U > U(0, \bar{p}, p)\). Then \(x^H\) solves the UMP at the price vector \(p\), reference price vector \(\bar{p}\) and income \(w = p.x^H\).
Proof: (i) Suppose that $x^H$ solves the EMP and $U(x^H, \bar{p}, p) > U$. Then continuity of $U$ implies that there is another bundle $x' = \alpha x^H$ that is close to $x^H$ ($0 < \alpha < 1$ and $\alpha$ close to 1) such that $U(x', \bar{p}, p) > U$ and $p.x' < p.x^H$. This contradicts the fact that $x^H$ solves the EMP.

(ii) Suppose that $x^M$ does not solve the EMP. Then there exists some bundle $x' \neq x^M$ that solves the EMP. It must be that $U(x', \bar{p}, p) \geq U(x^M, \bar{p}, p)$ and $p.x' < p.x^M \leq w$. Since preferences are locally non-satiated, for any $\varepsilon > 0$ there exists $y \in X$, $\|y - x'\| < \varepsilon$, such that $y >_{\bar{p}, p} x'$. Thus, $U(y, \bar{p}, p) > U(x', \bar{p}, p)$, yet $y$ is affordable, $p.y \leq w$. But this contradicts the fact that $x^M$ solve the UMP. Thus, $x^M$ must solve the EMP. Since $x^M$ solves the UMP and Walras’ Law holds (Proposition 2), hence, $p.x^M = w$.

(iii) Suppose that $x^H$ does not solve the UMP at $(\bar{p}, p, w)$ where $w = p.x^H$. Then there exists some $x' \neq x^H$ such that $U(x', \bar{p}, p) > U(x^H, \bar{p}, p)$ and $p.x' \leq p.x^H = w$; the assumption $\bar{U} > U(0, \bar{p}, p)$ ensures that $p.x^H > 0$. Now choose another bundle $y = tx'$, $0 < t < 1$ and $t$ close to 1. Clearly $p.y < p.x'$ and by continuity of $U$, $U(x', \bar{p}, p) > U(x^H, \bar{p}, p)$. But this contradicts the fact that $x^H$ solves the EMP. ■

Define the expenditure function

$$e(\bar{p}, p, U) = p.x^H(\bar{p}, p, U).$$

(4.1)

Proposition 5 implies the following important property for any $p \gg 0$, $w > 0$ and $\bar{U} > U(0, \bar{p}, p)$

$$e(\bar{p}, p, V(\bar{p}, p, w)) = w$$

and

$$V(\bar{p}, p, e(\bar{p}, p, U)) = U.$$  

(4.2)

Proposition 5 also implies the following relation between Hicksian and Marshallian demands.

$$h(\bar{p}, p, U) = x(\bar{p}, p, e(\bar{p}, p, U)),$$

$$x(\bar{p}, p, w) = h(\bar{p}, p, V(\bar{p}, p, w)).$$

(4.3)

### 4.1. The properties of the expenditure function

We now show that two of the central features of the expenditure function in classical consumer theory- homogeneity of degree 1 in $p$ and concavity in prices- do not hold in the presence of gain-loss utility. The failure of homogeneity has an obvious cause- prices also directly enter the utility function. To see the intuition for the failure of concavity in prices, consider Figure 4.1.

Consider first the absence of gain-loss utility. Suppose that the consumer solves the EMP in which the desired utility is $U$. At a price $\hat{p}$, the optimal Hicksian bundle is $\hat{x}$ and the expenditure is $e(\hat{p}, U) = \hat{p}\hat{x}$. The line $p\hat{x}$ shows the variation in expenditure as prices are allowed to vary but the consumption bundle is held fixed at $\hat{x}$. Clearly if the consumer is also allowed to choose the optimal bundle corresponding to each price, then the optimal expenditure, $e(p, U)$, will always be lower, hence, $p\hat{x} \geq e(p, U)$ for all $p$ (with equality at
Figure 4.1: Failure of concavity of the expenditure function under gain-loss utility.

\(\hat{p}\). Since this holds at any arbitrary initial price \(\hat{p}\), this implies that \(e(p, U)\) is concave, as shown.

Suppose that the initial price is \(\hat{p}\) and the new price is \(\bar{p} > \hat{p}\).

(i) In the absence of gain-loss utility the optimal Hicksian bundle is \(\bar{x}\) and the optimal expenditure is given by \(e(\bar{p}, U)\) (shown by the vertical distance \(AB\)) and the corresponding utility is \(u(\bar{x})\).

(ii) Now introduce gain-loss utility (\(\mu > 0\)). Let the reference price be \(\bar{p} = \bar{p}\). At a price \(\bar{p}\), there is no gain-loss utility, hence, the expenditure functions in the two cases coincide: \(e(\bar{p}, \bar{p}, U) = e(\bar{p}, U)\). At a price \(\bar{p} > \hat{p}\) and the consumption bundle \(\bar{x}\) (which is optimal in the absence of gain-loss utility), the utility in the presence of gain-loss utility is \(u(\bar{x}) - \theta \mu (\bar{p} - \bar{p}) < U\). Thus, \(\bar{x}\) does not solve the EMP under gain-loss utility when the required utility level is \(U\). Given monotonicity, one must increase the consumption bundle \(\bar{x}\) in at least one dimension in order to achieve the utility level \(U\) at a price \(\bar{p}\) when the reference price is \(\hat{p} = \bar{p}\). Suppose that the bundle \(x^*\) solves the EMP under gain-loss utility when the required utility is \(U\). Then we have \(e(\bar{p}, \bar{p}, U) = \bar{p}x^* > \bar{p}\bar{x} = e(\bar{p}, U)\). Indeed, nothing rules out the case shown in Figure 4.1 where \(e(\bar{p}, \bar{p}, U)\) lies above the line \(p\bar{x}\) and it is convex; we show a convex expenditure function for quasilinear preferences below.

Finally, we expect the expenditure function to be kinked at the reference price, \(\bar{p}\). The reason is that as \(p\) increases beyond \(\bar{p}\), the utility function is subject to loss aversion, hence, it is relatively more costly to achieve the same level of utility.

We now give a formal proof of these assertions.
Proposition 6: Suppose that $U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}$ represents continuous, locally non-satiated preferences, $\succeq_{p}$. The properties of the expenditure function $e(\bar{p}, p, U)$ are given by:

(i) $e(\bar{p}, p, U)$ is not homogenous of degree one in $p$.
(ii) $e(\bar{p}, p, U)$ is strictly increasing in $U$ and non-decreasing in $p_1, p_2$.
(iii) $e(\bar{p}, p, U)$ is not guaranteed to be concave in $p$.
(iv) $e(\bar{p}, p, U)$ is continuous in $p, U$ but kinked at the reference price, $\bar{p}$.

Proof: (i) Suppose that prices change from $p$ to $\alpha p$ with $\alpha > 0$ and $\alpha \neq 1$. Minimizing $p.x$ is equivalent to minimizing $\alpha p.x$. However, $U(x, \bar{p}, p) \neq U(x, \bar{p}, \alpha p)$. Hence, the constraint set changes and there is no guarantee that the solution is unchanged.

(ii) Proofs are standard so omitted.

(iii) In classical consumer theory the expenditure function is concave in prices. It is instructive to see where the proof breaks down under gain-loss utility. Suppose that there is no gain-loss utility ($\mu = 0$) and a utility level $U$ is to be attained in the EMP. Let $p, p'$ be two price vectors and let $p^c = \alpha p + (1 - \alpha)p'$, $\alpha \in [0, 1]$. Since the EMP has a solution under the stated conditions, suppose that the optimal consumption bundle that solves the EMP at the price $p^c$ is $x^c$, so

$$
eq \alpha e(p, U) + (1 - \alpha)e(p', U). \quad (4.4)$$

The last inequality follows from the fact that $x^c$ is not necessarily the optimal bundle at the prices $p, p'$, respectively. Hence, $e(p, U)$ is concave in $p$.

Now suppose that there is gain-loss utility ($\mu > 0$), the reference price vector is $\bar{p}$ and a utility level $U$ is to be attained in the EMP. The expenditure function is now denoted by $e(\bar{p}, p^c, U)$. The very last step in the proof above in (4.4) now breaks down. Let $p^c$ be the reference price (although this is not essential to the proof). Let $U(\succeq_{p^c, p'})$ be the indifference curve on which the optimal bundle $x^c$ lies under gain-loss utility when the price is $p^c$. Since $p^c$ is the reference price there is no gain-loss utility and the utility level from the bundle $x^c$ at the price $p^c$ is

$$U(\succeq_{p^c, p'}) = u(x^c).$$

By construction, it must be the case that one of $p, p'$ is higher than $p^c$. Without loss in generality let this be $p$ so that $p' < p^c < p$. Suppose that we can adjust the income of the
consumer in order to enable him to buy the bundle \( x^c \) at a price \( p \). Then the utility from this bundle is
\[
\hat{U} = u(x^c) - \mu \theta (p - p^c) < U(\geq p^c, p^c).
\]
Since monotonicity holds, we need to increase at least one component in the bundle \( x^c \) in order to achieve the level of utility \( U(\geq p^c, p^c) \) at the price vector \( p \). Thus, let \( x \gg x^c \) be the expenditure minimizing bundle at price \( p \) when the target utility is \( U(\geq p^c, p^c) \) (our assumptions guarantee the existence of such a bundle). Then
\[
e(p^c, p, U) = px
\]
Since \( x \gg x^c \), it follows that \( px > px^c \), or
\[
e(p^c, p, U) > px^c,
\]
hence, the last line in the proof of (4.4) breaks down and the expenditure function is no longer concave in prices.

(iv) This follows from the theorem of the maximum.

Henceforth, we revert to the more standard terminology of \( h(\bar{p}, p, U) = \{ x^H : x^H \text{ solves the EMP} \} \) as the set of Hicksian demands. Thus, the expenditure function is defined by
\[
e(\bar{p}, p, U) = p.h(\bar{p}, p, U).
\] (4.5)

4.2. Income and substitution effects under reference dependence

The solution to the EMP gives rise to Hicksian or compensated demands. The reason for using the term ‘compensated demands’ is that the constraint in the EMP binds (see Proposition 5(i)) so it ensures that as the set of prices changes, the utility stays fixed at \( U \). Hence, we trace out the set of demands at different prices that all give rise to the identical utility level \( U \), compensating the consumer for any changes in incomes.

Assume that preferences are strictly convex so that there is a unique solution. The more interesting case arises when the change in prices switches the domain- from gain to loss or loss to gains. This case is shown in Figure 4.2 when there is a unique solution to the EMP.

For simplicity we have assumed that \( p_2 = p_2^l \) and only the price of good 1 is varied. Suppose that the initial price vector is \( p_1 = (p_1^l, p_2^l) \), the final price vector is \( p_1 = (p_1^l, \bar{p}_2^l) \), and \( p_1^l < \bar{p}_1^l < p_1^l \); so there is a fall in the price of good 1 that switches the domain from losses to gains but income stays fixed at \( w \). Under gain-loss utility, an indifference curve, \( I^0 \), corresponding to the lower price vector \( p_1 \) is shown as the dotted curve. The Marshallian demand at the economic environment \( (\bar{p}, p_1, w) \) is given by \( x(\bar{p}, p_1, w) \) (see
point $D$), and the utility level attained at this bundle is $U(\geq \bar{p}, p_1)$. Using Proposition 5, the Hicksian bundle corresponding to point $D$ is labelled as $h(\bar{p}, p_1, \bar{U})$.\footnote{It is the solution to the EMP when a minimum utility $\bar{U}$ is to be achieved; $x^H \in \arg\min_{x \in X} p_h x$ subject to $U(x, \bar{p}, p_1) \geq \bar{U}$.}

A family of three continuous indifference curves, $II$, $I'\ I'$ and $I''\ I''$, corresponding to the high price vector $p_h$ is also shown. The original equilibrium at the economic environment $\left(\bar{p}, p_h, w\right)$ under gain-loss utility is shown at $A$; the corresponding indifference curve is $II$. Since $p^h_1 > p_1$, the consumer is in the domain of loss with respect to good 1, so the marginal utility of good 1 is relatively lower. Thus, the indifference curve labelled $II$ is relatively flatter as compared to $I^0\ I^0$.

The Hicksian demand at $p_h$ when the required utility is $\bar{U} = \bar{U}(\geq \bar{p}, p_h)$ is the solution to $x^H \in \arg\min_{x \in X} p_h x$ subject to $U(x, \bar{p}, p_1) \geq \bar{U}$.

The classical method of isolating income and substitution effects is to keep the price ratio fixed at the initial level, $\frac{p^h_1}{p_2}$, and increase the income of the consumer starting at point $A$ to ensure that the price line with slope $\frac{p^h_1}{p_2}$ is just tangent to the indifference curve on which the bundle $D$ lies. This is shown as the point $B$; the corresponding consumption bundle is shown as $\tilde{x}$. However, the indifference curve, $I^0\ I^0$ that gives utility $\bar{U}$ no longer applies at the price vector $p_h$ because indifference curves are indexed by the relation between $\bar{p}_1$ and $p_1$, which has now flipped domains. The relevant indifference curve that reflects the price vector $p_h$ and passes through point $B$ is $I'\ I'$. The utility level corresponding to $I'\ I'$ is $\bar{U}$. Fixing the consumption bundle $\tilde{x}$, one can show that...
\[ \bar{U} = U(\bar{x}, \overline{p}, p_h) < U = U(\bar{x}, \overline{p}, p_l) \] as follows.

\[ \overline{U} = u(\bar{x}) + \mu(p_1 - p'_1) \]
\[ \tilde{U} = u(\bar{x}) - \mu \theta(p^h_1 - \overline{p}_1) < U. \]

It follows that \( h(p_h, \overline{U}) \) does not belong to the set \( x^H \in \arg \min_{x \in X} p_h . x \) subject to \( U(x, \overline{p}, p_h) \geq \overline{U} \). Monotonicity implies that at the price vector \( p_h \), the solution lies to the north-east of the consumption bundle \( \bar{x} \) that gives the required higher utility \( \overline{U} \). A solution to this problem, \( h(\overline{p}, p_h, \overline{U}) \), is shown at the point \( C \), corresponding to a higher indifference curve \( I''I'' \) that achieves a utility level \( \overline{U} \) at prices \( p_h \).

In Figure 4.2, the move from \( A \) to \( C \) may be called the income effect and the move from \( C \) to \( D \) the substitution effect. An alternative decomposition that perhaps clarifies better the difference from the classical framework may be termed as follows.

\[
\begin{align*}
A \to B & \quad \text{Classical income effect} \\
B \to C & \quad \text{Reference income effect} \\
C \to D & \quad \text{Substitution effect}
\end{align*}
\]

### 4.3. Failure of the compensated law of demand

The Hicksian demand curve does not satisfy the compensated law of demand. It is first instructive to see how the proof works in the classical case without gain-loss utility.

**Proposition 7**: In the absence of gain-loss utility, the compensated law of demand holds, i.e., the Hicksian demand curves are non-increasing in own price.

**Proof**: Consider two price vectors \( p' \) and \( p'' \) and the corresponding unique levels of Hicksian demands \( h(p', \overline{U}) \) and \( h(p'', \overline{U}) \). By definition, each Hicksian bundle gives the least cost of achieving utility \( \overline{U} \) at the respective prices. Thus,

\[ p' . h(p', \overline{U}) \leq p' . h(p'', \overline{U}) \Rightarrow p' . (h(p', \overline{U}) - h(p'', \overline{U})) \leq 0. \]
\[ p'' . h(p'', \overline{U}) \leq p'' . h(p', \overline{U}) \Rightarrow -p'' . (h(p', \overline{U}) - h(p'', \overline{U})) \leq 0. \]

Add the two inequalities to get the compensated law of demand

\[ (p' - p'') \cdot (h(p', \overline{U}) - h(p'', \overline{U})) \leq 0. \quad (4.6) \]

If there is a change in the price of a single good, say good \( i \) alone, (4.6) can be written as

\[ (p'_i - p''_i) \cdot (h(p'_i, \overline{U}) - h(p''_i, \overline{U})) \leq 0. \]

Thus if the price of a good goes up then its compensated demand cannot go up. For compensated demands, own price effects are always non-positive; this does not hold for
Marshallian demands (e.g., Giffen goods are allowed). However, the proof of Proposition 7 breaks down under gain-loss utility. The reason is that achieving the utility level $\bar{U}$ at different price levels requires different consumption bundles in the presence of gain-loss utility, which is not the case in the classical framework.

In Figure 4.2, the substitution effect arising from a fall in the price of good 1 is reflected in the movement from $C$ to $D$; it involves a fall in the consumption of good 1, violating the law of compensated demand. If, however, point $C$ lies to the north-west of point $D$ then the compensated law of demand shall hold. There is no guarantee under gain-loss utility that it would. More formally, using the notation in Figure 4.2, in order for the law of compensating demand to hold, we require the following inequalities to hold.

\[
\begin{align*}
 p_l.h(p_l, p_l, \bar{U} (\succeq_{\bar{p}, p_l})) &\leq p_l.h(p_l, p_h, \bar{U} (\succeq_{\bar{p}, p_h})). \\
 p_h.h(p_l, p_h, \bar{U} (\succeq_{\bar{p}, p_h})) &\leq p_h.h(p_l, p_l, \bar{U} (\succeq_{\bar{p}, p_l})).
\end{align*}
\]

(4.7)

(4.8)

However, the second inequality (4.8) is violated because at a price $p_h$ the optimal Hicksian consumption bundle $C$ that achieves a utility level $\bar{U}$ is more expensive relative to the Hicksian bundle $B$ that achieves a utility level $\bar{U}$ at the price $p_l$.

Below we show that for quasilinear preferences, and for plausible values of parameters, the Hicksian demand curve is upward sloping (see Figure 4.3).

4.4. Properties of Hicksian demands under reference dependence

The next proposition gives two properties of the Hicksian demand curves that survive the introduction of gain-loss utility.

**Proposition 8**: Let $U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}$ represent continuous, locally non-satiated preferences, $\succeq_{\bar{p}, p}$. The properties of the Hicksian demand curve $h(\bar{p}, p, U)$ are:

(i) $h$ is not homogenous of degree zero in $p$.

(ii) If $\succeq_{\bar{p}, p}$ is convex then the set $h(\bar{p}, p, U)$ is convex. If $\succeq_{\bar{p}, p}$ is strictly convex then the set $h(\bar{p}, p, U)$ is a singleton.

Proof: (i) Follows from Proposition 6(i).

(ii) Let $\succeq_{\bar{p}, p}$ be convex and $x, x' \in h(\bar{p}, p, U)$, $x \neq x'$. By definition $p.x = p.x'$ and $U(x, \bar{p}, p) \geq U$ and $U(x', \bar{p}, p) \geq U$. Let $x^e = \alpha x + (1 - \alpha)x'$ and $\alpha \in [0, 1]$. $p.x^e = p.(\alpha x + (1 - \alpha)x') = p.x = p.x'$. By assumption of convex preferences, $U(x^e, \bar{p}, p) \geq U$. Thus, $x^e \in h(\bar{p}, p, U)$.

Now let $\succeq_{\bar{p}, p}$ be strictly convex, $x, x' \in h(\bar{p}, p, U)$, $x \neq x'$. By definition $p.x = p.x'$ and $U(x, \bar{p}, p) \geq U$ and $U(x', \bar{p}, p) \geq U$. Let $x^e = \alpha x + (1 - \alpha)x'$ and $\alpha \in (0, 1)$. By strict convexity, $x^e \succeq_{\bar{p}, p} x'$ so $U(x^e, \bar{p}, p) > U(x', \bar{p}, p)$. By continuity of $U$, for $t$ close to 1
(0 < t < 1 and t close to 1) such that \( U(tx^c, \bar{p}, p) > U \) and \( p.x' > p.tx^c \). This contradicts the fact that \( x' \) solves the EMP. Thus, there cannot be multiple solutions to the EMP. ■

4.5. A calculus based approach to the EMP and the modified Shephard’s lemma

Let \( U : X \times \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R} \) represent continuous, differentiable, locally non-satiated preferences, \( \succeq_{p,p} \) and let \( p \gg 0 \). The Lagrangian for the EMP is

\[
L = p.x - \lambda \left( U(x, \bar{p}, p) - \bar{U} \right),
\]

where \( \lambda \geq 0 \) is the Lagrangian multiplier and \( U(x, \bar{p}, p) = u(x) + \mu [g_1(\bar{p}, p)x_1 + g_2(\bar{p}, p)x_2] \).

The first order conditions evaluated at the optimal bundle \( h(\bar{p}, p, \bar{U}) \) are

\[
\frac{\partial L}{\partial x_i} = p_i - \lambda \left( u_i \left( h(\bar{p}, p, \bar{U}) \right) + \mu g_i \right) \geq 0, \quad i = 1, 2,
\]

(4.9)

where \( u_i = \frac{\partial u(h(\bar{p}, p, \bar{U}))}{\partial x_i} \).

\[
h_i \frac{\partial L}{\partial x_i} = 0, \quad i = 1, 2.
\]

From (4.9) we can eliminate \( \lambda \) to get

\[
\frac{p_i}{p_j} = \frac{u_i + \mu g_i}{u_j + \mu g_j}, \quad i \neq j.
\]

(4.10)

From the theorem of the maximum, we know that \( h_i \) is a continuous function of \( p_i \), but it might not be differentiable everywhere. Implicitly differentiating (4.9) with respect to \( p_i \) we get \( \frac{\partial h_i(p,p,\bar{U})}{\partial p_i} = \phi(\frac{\partial u_i(h(p,p,\bar{U}))}{\partial x_i},..) \). Let \( \{p_{1n}^-\} \) be a sequence of prices of good 1 that converges from the left to \( \bar{p}_1 \) and let \( \{p_{1n}^+\} \) be a sequence that converges from the right to \( \bar{p}_1 \).

\[
\lim_{p_{1n}^- - \bar{p}_1} \frac{\partial u_i(h(p,p,\bar{U}))}{\partial x_i} \neq \lim_{p_{1n}^+ - \bar{p}_1} \frac{\partial u_i(h(p,p,\bar{U}))}{\partial x_i}.
\]

Hence, \( \frac{\partial h_i(p,p,\bar{U})}{\partial p_i} \) is not continuous at \( \bar{p} \). Thus, we expect there to be a kink in the demand curve \( h_i(\bar{p}, p, \bar{U}) \) at the point \( \bar{p} \). We verify this below for specific functional forms.

We already know from Proposition 5 that the constraint in the EMP binds at the optimal solution, so \( U(h(\bar{p}, p, \bar{U}), \bar{p}, p) = \bar{U} \), or

\[
U(h(\bar{p}, p, \bar{U}), \bar{p}, p) = u \left( h(\bar{p}, p, \bar{U}) \right) + \mu \left[ g_1(\bar{p}_1, p_1)h_1(\bar{p}, p, \bar{U}) + g_2(\bar{p}_2, p_2)h_2(\bar{p}, p, \bar{U}) \right] = \bar{U}.
\]

(4.11)

Differentiating both sides of (4.11) with respect to \( p_i \) gives

\[
\sum_{j=1}^{2} \frac{\partial U \left( h(\bar{p}, p, \bar{U}), \bar{p}, p \right)}{\partial x_j} \frac{\partial h_j}{\partial p_i} + \frac{\partial U \left( h(\bar{p}, p, \bar{U}), \bar{p}, p \right)}{\partial p_i} = 0, \quad i = 1, 2.
\]

or

\[
\sum_{j=1}^{2} \left( \frac{\partial u \left( h(\bar{p}, p, \bar{U}) \right)}{\partial x_j} + \mu g_j \right) \frac{\partial h_j}{\partial p_i} + \mu \left[ h_i \frac{\partial g_i}{\partial p_i} \right] = 0, \quad i = 1, 2.
\]

(4.12)
Example 9 (Quasilinear Preferences): Suppose that
\[ u(x) = x_1 + \ln x_2. \]

From (4.9), the first order conditions for an interior solution are:
\[
\begin{cases}
p_1 - \lambda (1 + \mu g_1) = 0 & \Rightarrow \lambda = \frac{p_1}{1 + \mu g_1} \\
p_2 - \lambda \left( \frac{1}{x_2} + \mu g_2 \right) = 0 & \Rightarrow \lambda = \frac{p_2}{\frac{1}{x_2} + \mu g_2}.
\end{cases}
\] (4.13)

It follows that \( \frac{p_1}{1 + \mu g_1} = \frac{p_2}{\frac{1}{x_2} + \mu g_2} \). Solving out for the unique Hicksian demand for good 2, we get
\[
h_2(\vec{p}, p, U) = \frac{p_1}{p_2 (1 + \mu g_1) - \mu p_1 g_2}. \tag{4.14}
\]

From Proposition 5, the constraint binds when evaluated at the optimal solution \( h_1 = h_1(\vec{p}, p, U) \), \( h_2 = h_2(\vec{p}, p, U) \), so
\[
h_1 + \ln h_2 + \mu [g_1(\vec{p}, p)h_1 + g_2(\vec{p}, p)h_2] = U
\]

Solving out for \( h_1 \) we get
\[
h_1(\vec{p}, p, U) = \frac{1}{1 + \mu g_1} \left( U - \ln h_2(\vec{p}, p, U) - \mu g_2 h_2(\vec{p}, p, U) \right). \tag{4.15}
\]

Substituting (4.14) and (4.15) in (4.5) we get the expenditure function
\[
e(\vec{p}, p, U) = \frac{p_1 (U - \ln h_2(\vec{p}, p, U))}{(1 + \mu g_1)} + h_2(\vec{p}, p, U) \left( p_2 - \frac{p_1 \mu g_2}{1 + \mu g_1} \right), \tag{4.16}
\]

where \( h_2(\vec{p}, p, U) \) is given by (4.14). Also note that \( \lambda \) is the shadow price of achieving an extra unit of utility.
\[
\frac{\partial e(\vec{p}, p, U)}{\partial U} = \lambda. \tag{4.17}
\]

A tractable special case arises when \( p_2 = \vec{p}_2 \) so that \( g_2(\vec{p}_2, \vec{p}_2) = 0 \). We summarize the solution in this case:
\[
\begin{cases}
h_1(\vec{p}, p, U) = \frac{1}{1 + \mu g_1} (U - \ln (p_1/\vec{p}_2) + \ln (1 + \mu g_1)) \\
h_2(\vec{p}, p, U) = \frac{p_1/\vec{p}_2}{1 + \mu g_1} \\
e(\vec{p}, p, U) = \frac{p_1/\vec{p}_2}{1 + \mu g_1} \left( 1 + U - \ln (p_1/\vec{p}_2) + \ln (1 + \mu g_1) \right).
\end{cases} \tag{4.18}
\]

To ensure a sensible solution we require the condition
\[
1 + \mu g_1 > 0. \tag{4.19}
\]

Let us plot \( h_1(\vec{p}, p, U) \) and \( e(\vec{p}, p, U) \) for the parameter values \( U = 6, \vec{p}_1 = 2, \vec{p}_2 = p_2 = 1, \theta = 2.5, \mu = 0.2 \). The plot is shown in Figure 4.3. The continuous expenditure function
Figure 4.3: Plot of the expenditure function (continuous curve) and the Hicksian demand for good 1 (dashed curve) when the price of good 1 alone varies.

is convex and kinked at the reference price \( \bar{p}_1 = 2 \), as shown in Figure 4.1. The dashed Hicksian demand curve slopes downward over an initial range of prices but then becomes positively sloped. Both curves are defined only up to an upper limit \( p_1 \), which ensures that the inequality (4.19) holds.

The next example verifies that there is a kink in the Hicksian demand curves at \( \bar{p} \).

**Example 10** (Kinks in the Hicksian demand curves at \( p \). Example 9 continued...): Consider the special case \( p_2 = \bar{p}_2 \) so that the solution is given in (4.18). The Hicksian demand curves and the expenditure functions are continuous in \( p_1 \) but there is a kink in the Hicksian demand curves for both goods at a price \( p_1 = \bar{p}_1 \). To see this, first write the derivatives of the Hicksian demands with respect to \( p_1 \).

\[
\begin{align*}
\frac{\partial h_1}{\partial p_1} &= \frac{1}{(1+\mu g_1)} \left( -\frac{1}{p_1} + \frac{1}{(1+\mu g_1)} \frac{\partial g_1}{\partial p_1} \right) - h_1(p, \bar{p}, \bar{U}) \mu \frac{\partial g_1}{\partial p_1} \\
\frac{\partial h_2}{\partial p_1} &= \frac{1}{p_2(1+\mu g_1)^2} \left( 1 + \mu g_1 - p_1 \mu \frac{\partial g_1}{\partial p_1} \right)
\end{align*}
\]  

(4.20)

Let \( \{p_1^-\} \) be a sequence of prices of good 1 that converges from the left to \( \bar{p}_1 \) and let \( \{p_1^+\} \) be a sequence that converges from the right to \( \bar{p}_1 \). Using (2.3), we get

\[
\lim_{p_1^+ \to \bar{p}_1} \frac{\partial h_1}{\partial p_1} = \left( -\frac{1}{\bar{p}_1} - \mu \left( 1 - (\bar{U} - \ln (\bar{p}_1/\bar{p}_2)) \right) \right) + h_1(\bar{p}, \bar{p}, \bar{U}) \mu
\]

\[
\neq \lim_{p_1^- \to \bar{p}_1} \frac{\partial h_1}{\partial p_1} = \left( -\frac{1}{\bar{p}_1} - \theta \mu \left( 1 - (\bar{U} - \ln (\bar{p}_1/\bar{p}_2)) \right) \right) + h_1(\bar{p}, \bar{p}, \bar{U}) \theta \mu.
\]
\[
\lim_{p_i \to p_i} \frac{\partial h_2}{\partial p_1} = \frac{1}{p_2} (1 + \bar{p}_1 \mu) \neq \lim_{p_i \to p_i} \frac{\partial h_2}{\partial p_1} = \frac{1}{p_2} (1 + \theta \bar{p}_1 \mu).
\]
Hence, there is a kink in the Hicksian demand curves for both goods at a price \( p_1 = \bar{p}_1 \).

**Proposition 9** (*Modified Shephard’s Lemma*): Let \( U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R} \) represent continuous, locally non-satiated and strictly convex preferences, \( \succeq_{p,p} \). For any vector \((\bar{p},p,\bar{U})\), we have
\[
\frac{\partial e(\bar{p},p,\bar{U})}{\partial p_i} = h_i(\bar{p},p,\bar{U}) \left( 1 - \mu \lambda \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} \right),
\]
where \( \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} \in \{-1,-\theta\} \).

**Proof**: From (4.5) we have \( e(\bar{p},p,\bar{U}) = p_i h_i(\bar{p},p,\bar{U}) \). Differentiating both sides with respect to \( p_i \)
\[
\frac{\partial e(\bar{p},p,\bar{U})}{\partial p_i} = h_i(\bar{p},p,\bar{U}) + \sum_{j=1}^{2} p_j \frac{\partial h_j}{\partial p_i}.
\]
Substituting \( p_j = \lambda \left( \frac{\partial u(h(\bar{p},p,\bar{U}))}{\partial x_j} + \mu g_j \right) \) from (4.9) in (4.22) we get
\[
\frac{\partial e(\bar{p},p,\bar{U})}{\partial p_i} = h_i(\bar{p},p,\bar{U}) + \lambda \sum_{j=1}^{2} \left( \frac{\partial u(h(\bar{p},p,\bar{U}))}{\partial x_j} + \mu g_j \right) \frac{\partial h_j}{\partial p_i}.
\]
Substituting (4.12) in (4.23) we get
\[
\frac{\partial e(\bar{p},p,\bar{U})}{\partial p_i} = h_i(\bar{p},p,\bar{U}) - \lambda \mu h_i(\bar{p},p,\bar{U}) \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} \left( 1 - \mu \lambda \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} \right) .
\]  

In the special case where gain-loss utility is absent \((\mu = 0)\) we get the classical Shephard’s Lemma, \( \frac{\partial e(\bar{p},p,\bar{U})}{\partial p_i} = h_i(\bar{p},p,\bar{U}) \). However, in the presence of gain-loss utility, there is a correction to the Hicksian demand by a factor \( \left( 1 - \mu \lambda \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} \right) > 0 \) (recall that \( \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} |_{p_i < \bar{p}_i} = -1 \) and \( \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} |_{p_i > \bar{p}_i} = -\theta \)). In the classical case, there are no first order effects of changes in prices on expenditure (second term in (4.24) is zero when \( \mu = 0 \)). However, in the presence of gain-loss utility there are first order effects on the utility function of a change in prices which alter the expenditure minimizing consumption bundle.

The term \( \left( 1 - \mu \lambda \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} \right) \) plays a key role in several results below, so we define it formally.

**Definition 5** (*Behavioral multiplier*): The behavioral multiplier is defined as \( 1 - \mu \lambda \frac{\partial g_i(\bar{p},p_i)}{\partial p_i} \).
The intuition behind the behavioral multiplier is as follows. Recall the definition of the utility function in (4.11). The term $\mu \frac{\partial g_i(p, p_i)}{\partial p_i}$ is the direct effect on $U$ of a unit increase in $p_i$. By definition, $\lambda = \frac{\partial e(p, p, U)}{\partial p}$ is the shadow cost of achieving an extra unit of utility (see for instance (4.17) for the case of quasilinear preferences). Since $\frac{\partial g_i(p, p_i)}{\partial p_i} < 0$, the term $-\mu \lambda \frac{\partial g_i(p, p_i)}{\partial p_i}$ gives the shadow cost of a unit increase in prices that works through the gain-loss channel. For $h_i$ units of demand the total shadow cost of an increase in price is $-\mu \lambda \frac{\partial g_i(p, p_i)}{\partial p_i} h_i$.

By contrast in the classical case, the multiplier is 1 because $\frac{\partial e(p, p, U)}{\partial p_i} = h_i(p, p_i)$; this gives the direct extra cost from a unit increase in prices. In the presence of gain-loss utility the total marginal cost of an increase in prices is $h_i(p, p, U) - \mu \lambda \frac{\partial g_i(p, p_i)}{\partial p_i} h_i(p, p, U) \left(1 - \mu \lambda \frac{\partial g_i(p, p_i)}{\partial p_i}\right)$, which is the RHS of (4.21). For this reason, we find it convenient to term $\left(1 - \mu \lambda \frac{\partial g_i(p, p_i)}{\partial p_i}\right)$ as the behavioral multiplier.

**Example 11** (Verifying Shepard’s lemma for quasilinear preferences): Let us verify $\frac{\partial e(p, p, U)}{\partial p_2} = h_2(p, p, U) \left(1 - \mu \lambda \frac{\partial g_2(p, p_2)}{\partial p_2}\right)$ for the case of Example 10 when $p_2 = p_2$. The relevant Hicksian demands and the expenditure function is given by (4.18). Differentiating the expenditure function with respect to $p_2$ we get:

$$\frac{\partial e(p, p, U)}{\partial p_2} = \frac{p_1/p_2}{1 + \mu g_1}.$$  

Noting that $g_2(p, p_2) = 0$ and $\frac{\partial g_2(p, p_2)}{\partial p_2} = 0$, we have

$$h_2(p, p, U) \left(1 - \mu \lambda \frac{\partial g_2(p, p_2)}{\partial p_2}\right) = \frac{p_1/p_2}{1 + \mu g_1},$$

which confirms the formula when partials with respect to $p_2$ are taken. Differentiating the expenditure function with respect to $p_1$:

$$\frac{\partial e(p, p, U)}{\partial p_1} = \frac{1}{1 + \mu g_1} \left[U - \ln \left(p_1/p_2\right) + \ln \left(1 + \mu g_1\right)\right] \left[1 - \mu \frac{1}{1 + \mu g_1} \frac{\partial g_1(p, p_1)}{\partial p_1}\right]$$

$$= h_1(p, p, U) \left(1 - \mu \lambda \frac{\partial g_1(p, p_1)}{\partial p_1}\right),$$

as required by Proposition 9.

**4.6. Asymmetry of Hicksian cross partial derivatives**

From Proposition 6 we know that $e(p, p, U)$ is not necessarily a concave function of $p$. Consider the matrix of second order partial derivatives

$$D_{pq}e(p, p, U) = \left(\frac{\partial^2 e(p, p, U)}{\partial p_j \partial p_i}\right)_{2 \times 2} = \left(\frac{\partial h_i}{\partial p_j} \left(1 - \lambda \mu \frac{\partial g_i}{\partial p_i}\right) - h_i \mu \frac{\partial g_i}{\partial p_i} \frac{\partial \lambda}{\partial p_i} \frac{\partial \lambda}{\partial p_j}\right)_{2 \times 2}.$$  

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From a well known result in calculus, if all second order partial derivatives \( \frac{\partial^2 e(p, p, U)}{\partial p_j \partial p_i} \) are continuous at a point \( p \) then the matrix \( D_{pp}e(p, p, U) \) is symmetric, i.e., for \( i \neq j \) we have \( \frac{\partial^2 e(p, p, U)}{\partial p_j \partial p_i} = \frac{\partial^2 e(p, p, U)}{\partial p_i \partial p_j} \). However, we already know (see Example 10) that \( \frac{\partial h_i}{\partial p_j} \) is not continuous at \( p \) because of the kink in \( h_i \) at \( p \); hence \( \frac{\partial^2 e(p, p, U)}{\partial p_j \partial p_i} \) is also not continuous at \( p \). This is illustrated in Figure 4.4. Hence, the symmetry property of the expenditure function does not hold at \( p \). In empirical research the symmetry of the Hicksian cross partial derivatives cannot then be used as an identifying restriction. Household data rejects symmetry of the Slutsky matrix (Blundell et al., 1993; Browning and Meghir, 1991). Browning and Chiappori (1998) report support of Slutsky symmetry from singles within a household but rejection of symmetry for couples in 2 member households.

**Example 12** (Asymmetry of cross partial derivatives of Hicksian demands. Example 9 continued...): Consider the special case \( p_2 = \bar{p}_2 \) so that the solution is given in (4.18). A simple calculation gives the following when evaluated at \( p_2 = \bar{p}_2 \)

\[
\begin{align*}
\frac{\partial h_1}{\partial p_2} &= \frac{1}{\bar{p}_2(1+\mu g_1)} \\
\frac{\partial h_2}{\partial p_1} &= \frac{1}{\bar{p}_2(1+\mu g_1)^2} \left( 1 + \mu g_1 - p_1 \mu \frac{\partial g_1}{\partial p_1} \right) = \frac{1}{\bar{p}_2(1+\mu g_1)} \frac{\partial h_1}{\partial p_2} + \frac{1}{\bar{p}_2(1+\mu g_1)^2} \left( \mu g_1 - p_1 \mu \frac{\partial g_1}{\partial p_1} \right).
\end{align*}
\]

Clearly \( \frac{\partial h_1}{\partial p_2} \bigg|_{p_2=\bar{p}_2} \neq \frac{\partial h_2}{\partial p_1} \bigg|_{p_2=\bar{p}_2} \), which violates the symmetry of cross partial derivatives of Hicksian demands. In the absence of gain-loss utility (\( \mu = 0 \)) we have \( \frac{\partial h_1}{\partial p_2} = \frac{\partial h_2}{\partial p_1} = \frac{1}{\bar{p}_2} \). Thus, it is gain-loss utility that breaks down the symmetry argument.

To summarize: In the classical case where gain-loss utility is absent (\( \mu = 0 \)), \( \frac{\partial h_i}{\partial p_j} \) is continuous at \( p \) and the cross partials of the compensated are symmetric, i.e., \( \frac{\partial^2 h_i(p, p, U)}{\partial p_j \partial p_i} = \)
\( \frac{\partial h_2(p, p, U)}{\partial p_1} \). However, in the presence of gain-loss utility there is no presumption that symmetry holds. Commenting on the classical result of the symmetry of cross price derivatives of the compensated demand curves when \( \mu = 0 \), Mas-Colell et al. (1995, p. 70) write "The symmetry of \( D_p h(p, U) \) is an unexpected property. Symmetry is not easy to interpret in plain economic terms. As emphasized by Samuelson (1947), it is a property just beyond what one would derive without the help of mathematics." However, lack of symmetry in the presence of gain-loss utility is not an unexpected property but it would not be possible to state it without the help of mathematics.

4.7. The link between Marshallian and Hicksian demands (modified Slutsky equation)

Let us now derive the analogue of the Slutsky equation when a consumer faces the price-income vector \((p, w)\) and wishes to achieve at least the utility \(U\). This enables us to derive a link between the observable Marshallian demands and the unobservable Hicksian demands.

**Proposition 10** (Modified Slutsky equation): Let \( U : X \times \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R} \) represent continuous, differentiable, locally non-satiated preferences, \( \succeq_{p,p} \), and let \( p \gg 0 \). Then the link between the Marshallian and the Hicksian demands is given by the following

\[
\frac{\partial h_i(p, p, U)}{\partial p_j} = \frac{\partial x_i(p, p, w)}{\partial p_j} + \frac{\partial x_i(p, p, w)}{\partial w} x_j(p, p, w) \left( 1 - \lambda \mu \frac{\partial g_j}{\partial p_j} \right),
\]

where \( \frac{\partial g_j}{\partial p_j} \in \{-1, -\theta\} \) (see (2.3)).

**Proof:** From (4.5) we have \( h(p, p, U) = x(p, p, e(p, p, U)) \). Let \( w = e(p, p, U) \). In particular, for any commodity \( i = 1, 2 \) we have

\[
h_i(p, p, U) = x_i(p, p, w).
\]

Differentiating both sides with respect to \( p_j, j = 1, 2 \) we get

\[
\frac{\partial h_i(p, p, U)}{\partial p_j} = \frac{\partial x_i(p, p, w)}{\partial p_j} + \frac{\partial x_i(p, p, w)}{\partial w} \frac{\partial e(p, p, U)}{\partial p_j}.
\]

From (4.21) we have \( \frac{\partial e(p, p, U)}{\partial p_j} = h_j(p, p, U) \left( 1 - \mu \lambda \frac{\partial g_j(p, p)}{\partial p_j} \right) \). Substitute in (4.27)

\[
\frac{\partial h_i(p, p, U)}{\partial p_j} = \frac{\partial x_i(p, p, w)}{\partial p_j} + \frac{\partial x_i(p, p, w)}{\partial w} \left( h_j(p, p, U) \left( 1 - \mu \lambda \frac{\partial g_j(p, p)}{\partial p_j} \right) \right)
\]

Substitute (4.26) in (4.28) we get

\[
\frac{\partial h_i(p, p, U)}{\partial p_j} = \frac{\partial x_i(p, p, w)}{\partial p_j} + \frac{\partial x_i(p, p, w)}{\partial w} x_j(p, p, w) \left( 1 - \lambda \mu \frac{\partial g_j}{\partial p_j} \right),
\]

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where \( \frac{\partial g_j}{\partial p_j} \in \{-1, -\theta\} \) is given by (2.3) and \( 1 - \lambda \mu \frac{\partial g_i}{\partial p_j} > 0 \). \( \blacksquare \)

From (4.25), in the absence of gain-loss utility \( (\mu = 0) \), we have the classical Slutsky equation.

\[
\frac{\partial h_i(p, U)}{\partial p_j} = \frac{\partial x_i(p, w)}{\partial p_j} + \frac{\partial x_i(p, w)}{\partial w} x_j(p, w). \tag{4.30}
\]

Suppose that all goods are normal goods. By definition, the two demand curves are equal at the price vector \( p \) because from (4.3) we have \( h(\bar{p}, p, U) = x(\bar{p}, p, e(\bar{p}, p, U)) \). For normal goods, \( \frac{\partial x_i(\bar{p}, p, w)}{\partial w} > 0 \). Thus, we have from (4.25) that the Hicksian demand curve is relatively steeper as compared to the Marshallian demand curve (this holds in the presence and in the absence of gain-loss utility).

Using Proposition 10, in Figure 4.5 we show the relation between the Hicksian and Marshallian demand curves in the presence of gain-loss utility. \( x_1(\bar{p}, p, w) \) and \( h_1(\bar{p}, p, U) \).

The Hicksian and Marshallian demand curves intersect at the current price (this follows from (4.3); this is shown as point \( A \)). Gain-loss utility creates a kink in the Marshallian and Hicksian demand curves (points \( B \) and \( C \)).

4.8. The analogue of Roy's identity under gain-loss utility

**Proposition 11**: Let \( U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R} \) represent continuous, locally non-satiated and strictly convex preferences, \( \succeq_{p, p} \). Then the Marshallian demand curves can be ex-
pressed as the formula

\[ x_j (\overline{p}, p, w) = \frac{1}{\left(1 - \mu \lambda \frac{\partial g_j (\overline{p}, p_j)}{\partial p_j}\right)} \left(- \frac{\partial V(\overline{p}, p, w)}{\partial p_j} - \frac{\partial V(\overline{p}, p, w)}{\partial w}\right), \ j = 1, 2. \]  

(4.31)

Proof: Let \( V(\overline{p}, p, w) = U \). From (4.2) we have \( V(\overline{p}, p, e(\overline{p}, p, U)) = U \). Differentiating both sides with respect to \( p_j \),

\[ \frac{\partial V(\overline{p}, p, e(\overline{p}, p, U))}{\partial p_j} + \frac{\partial V(\overline{p}, p, e(\overline{p}, p, U))}{\partial w} \left( h_j (p, p, U) \left(1 - \mu \lambda \frac{\partial g_j (\overline{p}, p_j)}{\partial p_j}\right) \right) = 0. \]  

(4.32)

Using (4.21) to substitute out \( \frac{\partial e(\overline{p}, p, U)}{\partial p_j} \), we get

\[ \frac{\partial V(\overline{p}, p, e(\overline{p}, p, U))}{\partial p_j} + \frac{\partial V(\overline{p}, p, e(\overline{p}, p, U))}{\partial w} \left( x_j (\overline{p}, p, w) \left(1 - \mu \lambda \frac{\partial g_j (\overline{p}, p_j)}{\partial p_j}\right) \right) = 0 \]  

(4.33)

Simplifying, we get

\[ x_j (\overline{p}, p, w) = \frac{1}{\left(1 - \mu \lambda \frac{\partial g_j (\overline{p}, p_j)}{\partial p_j}\right)} \left(- \frac{\partial V(\overline{p}, p, w)}{\partial p_j} - \frac{\partial V(\overline{p}, p, w)}{\partial w}\right), \ j = 1, 2, \]

where \( \frac{\partial g_j (\overline{p}, p_j)}{\partial p_j} \in \{-1, -\theta\} \).  

4.9. Properties of the transformed Hicksian demand

From Proposition 9, we have \( \frac{\partial e(\overline{p}, p, U)}{\partial p_i} = h_i^T (\overline{p}, p, U) \) where

\[ h_i^T (\overline{p}, p, U) = h_i (\overline{p}, p, U) \left(1 - \mu \lambda \frac{\partial g_i (\overline{p}, p_i)}{\partial p_i}\right) \geq 0. \]  

(4.34)

\( \frac{\partial g_i (\overline{p}, p_i)}{\partial p_i} \in \{0, i\} \) and \( \lambda = \lambda (\overline{p}, p, U) \) is the Lagrangian multiplier in the EMP. We call \( h_i^T \) the ‘transformed’ Hicksian demand function for good \( i \) due to the multiplication by the behavioral multiplier (Definition 5).

From Proposition 6, we know that \( e(\overline{p}, p, U) \) is not necessarily a concave function of \( p \), hence,

\[ \frac{\partial^2 e}{\partial p_i^2} = \frac{\partial h_i^T}{\partial p_i} \leq 0. \]  

(4.35)

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Rewriting (4.34) in the domain of gains and losses, we get:

\[ h^T_i(\bar{p}, p, U) = \begin{cases} 
(1 + \mu \lambda^+(\bar{p}, p, U)) h^+_i(\bar{p}, p, U) & \text{if } \bar{p}_i \geq p_i \\
(1 + \theta \mu \lambda^-(\bar{p}, p, U)) h^-_i(\bar{p}, p, U) & \text{if } \bar{p}_i < p_i 
\end{cases} \]  

(4.36)

where superscripts + and − indicate the values of the relevant variables in the domain of gains and losses. This applies to the endogenous variables, λ and h. We know from the theorem of the maximum that λ and h are continuous. However, the transformed Hicksian demand curve is not continuous. To see this, let \( \{p^-_i\} \) be a sequence of prices of good 1 that converges from the left to \( \bar{p}_i \) and let \( \{p^+_i\} \) be a sequence that converges from the right to \( \bar{p}_i \). Using (4.36), we get

\[
\lim_{p^-_i \to \bar{p}_i} h^T_i(\bar{p}, p, U) = (1 + \mu \lambda) h_i \neq \lim_{p^+_i \to \bar{p}_i} h^T_i(\bar{p}, p, U) = (1 + \theta \mu \lambda) h_i.
\]

However, \( h^T_i(\bar{p}, p, U) \) is piecewise continuous over \([0, \bar{p}_i] \cup (\bar{p}_i, p^+_i)\) where \( p^+_i \) is some upper bound on prices; this allows one to conduct welfare analysis using \( h^T_i \). Since \( \left(1 - \mu \lambda \frac{\partial g_i(\bar{p}, p)}{\partial p_i}\right) > 0 \) it follows that

\[ h^T_i(\bar{p}, p, U) > h_i(\bar{p}, p, U). \]

Thus, the transformed Hicksian demand curve lies everywhere to the right of the standard Hicksian demand curve under gain-loss utility. From (4.9) we have \( \lambda = \frac{p_i}{u_i + \mu g_i} \) so

\[ \lambda = \begin{cases} 
\lambda^+ = \frac{p_i}{u_i + \mu g_i} & \text{if } \bar{p}_1 \geq p_1 \\
\lambda^- = \frac{p_i}{u_i - \theta \mu (p_i - \bar{p}_i)} & \text{if } \bar{p}_1 < p_1
\end{cases}
\]

(4.38)

where \( \lambda \) is a continuous function of the parameters. It is clear that \( \frac{\partial \lambda^-}{\partial \theta} > 0 \). However, \( \lambda \) is not differentiable at \( \bar{p} = (\bar{p}_1, \bar{p}_2) \). Provided that the denominator in (4.38) is non-zero, \( \lim_{p_i \to 0} \lambda = 0 \); since \( \lambda = \frac{\partial g_i}{\partial \theta} \), thus, if a good is free it adds no marginal costs to achieve an extra unit of utility \( U \). Implicitly differentiating (4.38) and noting that \( u_i - \theta \mu (p_i - \bar{p}_i) \geq 0 \) (because \( \lambda \geq 0 \)) we get

\[
\frac{\partial \lambda^+}{\partial p_i} = \frac{[u_i + \mu g_i]}{(u_i + \mu g_i)^2} > 0; \quad \frac{\partial \lambda^-}{\partial p_i} = \frac{[u_i - \theta \mu (p_i - \bar{p}_i)] + \theta \mu p_i}{(u_i - \theta \mu (p_i - \bar{p}_i))^2} > 0.
\]

(4.39)

\[
\lim_{p_i \to \bar{p}_i} \frac{\partial \lambda^+}{\partial p_i} = \frac{u_i + \mu \bar{p}_i}{u_i^2} < \lim_{p_i \to \bar{p}_i} \frac{\partial \lambda^-}{\partial p_i} = \frac{u_i + \theta \mu \bar{p}_i}{u_i^2}
\]

The variation of the behavioral multiplier \( \left(1 - \mu \lambda \frac{\partial g_i(\bar{p}, p)}{\partial p_i}\right) \) in (4.34) with respect to price is determined by the behavior of \( \lambda \) outlined above. Thus, as \( p_i \) increases, the behavioral multiplier becomes larger and there is a discontinuous jump as \( p_i \) crosses the threshold of \( \bar{p}_i \).

These observations lead to the shape of \( h^T_i \) shown in Figure 4.6. In the figure we show the Hicksian demand curve \( h_i(\bar{p}, p, U) \) to be the continuous kinked downward sloping curve
although there is no requirement that it be downward sloping under gain-loss utility. We have shown the transformed Hicksian curve to be downward sloping, although from (4.35), it is not required to be downward sloping (unless $h_i(p, p, U)$ is downward sloping). $h^T_i$ is not continuous at the reference price, as shown above, but it is piecewise continuous.

**Example 13 (Upward sloping transformed Hicksian demand for quasilinear preferences):**  
For the quasilinear case and $p_2 = p_2' = 1$ we can use (4.13), (4.18) to write:

$$
\begin{align*}
\lambda &= \frac{p_1}{(1 + \mu p_1)} \\
\frac{1}{1 + \mu p_1} h_1(p, p, U) &= U - \ln (p_1) + \ln (1 + \mu p_1) \\
\text{where } h^+_1 \text{ and } h^-_1 \text{ denote, respectively, the Hicksian demands when the consumer is the} \\
\text{domain of gains } (\bar{p}_1 \geq p_1) \text{ and losses } (\bar{p}_1 < p_1). \text{ Substituting the values of } h^+_1 \text{ and } h^-_1 \text{ we get}
\end{align*}
$$

Using (2.2), (2.3), (4.36) we get the transformed Hicksian demand:

$$
\begin{align*}
h^T_1(p, p, U^1) &= \begin{cases} 
\left(1 + \mu \frac{p_1}{(1 + \mu (\bar{p}_1 - p_1))}\right) h^+_1(p, p, U^1) & \text{if } \bar{p}_1 \geq p_1 \\
\left(1 + \theta \mu \frac{p_1}{(1 + \mu (-\theta (p_1 - \bar{p}_1)))}\right) h^-_1(p, p, U^1) & \text{if } \bar{p}_1 < p_1
\end{cases} \\
\end{align*}
$$

where $h^+_1$ and $h^-_1$ denote, respectively, the Hicksian demands when the consumer is the domain of gains ($\bar{p}_1 \geq p_1$) and losses ($\bar{p}_1 < p_1$). Substituting the values of $h^+_1$ and $h^-_1$ we get

$$
\begin{align*}
h^T_1(p, p, U^1) &= \begin{cases} 
\frac{1 + \mu p_1}{1 + \mu (\bar{p}_1 - p_1)} [U^1 - \ln (p_1) + \ln (1 + \mu (\bar{p}_1 - p_1))] & \text{if } \bar{p}_1 \geq p_1 \\
\frac{1 - \mu \theta (p_1 - \bar{p}_1)}{1 - (\mu (p_1 - \bar{p}_1))} [U^1 - \ln (p_1) + \ln (1 - \mu \theta (p_1 - \bar{p}_1))] & \text{if } \bar{p}_1 < p_1
\end{cases}
\end{align*}
$$

Figure 4.6: The Hicksian demand curve, $h_i$, for good $i$ and the transformed Hicksian demand curve, $h^T_i$, for good $i$
Figure 4.7: Plot of $h_1^T(\bar{p}, p, U^1)$ against $p_1$ for the parameter values: $\bar{U} = 6$, $\bar{p}_1 = 2$, $\bar{p}_2 = p_2 = 1$, $\mu = 0.2$. Two values of $\theta = 2.5$, 4.0 are used.

Figure 4.7 shows a plot of $h_1^T(\bar{p}, p, U^1)$ for the same parameter values as in Figure 4.3: $\bar{U} = 6$, $\bar{p}_1 = 2$, $\bar{p}_2 = p_2 = 1$, $\mu = 0.2$. We have chosen two values of loss aversion: $\theta = 2.5$ and $\theta = 4.0$. In comparing Figure 4.3 and Figure 4.7, note that while the Hicksian demand curve is simply kinked at the reference price of $\bar{p}_1 = 2$, the transformed Hicksian demand is discontinuous at the reference price. From (4.39), $\lambda$ is increasing in $p_1$, thus, the sign of $\frac{\partial h_1^T}{\partial p_1}$ is determined by the sign of $\frac{\partial \lambda}{\partial p_1}$. From Figure 4.3, $\frac{\partial \lambda}{\partial p_1} > 0$ for most of the price range, thus, in Figure 4.7 we have $\frac{\partial h_1^T}{\partial p_1} > 0$. The simulations show that as $\theta$ increases from 2.5 to 4.0, and for any fixed $p_1 > \bar{p}_1 = 2$, $h_1^T$ is higher under greater loss aversion (the vertical lines in Figure 4.7 indicate the highest price that allows for a positive value of $1 - \mu \theta (p_1 - \bar{p}_1)$).

5. Consumer welfare

Let $U : X \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}$ represent continuous, locally non-satiated preferences, $\succeq_{p,p}$. The economic environment in the UMP is captured by the vector $(\bar{p}, p, w)$. In this Section, we ask the question: What is the effect on consumer welfare when prices change? Suppose that the initial economic environment is given by $(\bar{p}, p^0, w)$ and the final economic environment by $(\bar{p}, p^1, w)$. How does this impact on the welfare of the consumer? We abstract away from changes in $\bar{p}$ and $w$, hence, in the spirit of comparative statics in the classical theory of the consumer, we are concerned with short-run changes.
5.1. Equivalent and compensating variation: Basic concepts

Let $V(p, p, w)$ be the indirect utility function derived from the preferences $\succeq_{p,p}$. Then, the consumer is better off after the change in prices if

$$V(p, p, w) < V(p, p, w).$$

(5.1)

Utility has ordinal significance. However, we may also express welfare changes in money terms by using a money metric utility function as follows. The function $e(p, p^i, V(p, p, w))$ denotes the income required to achieve utility $V(p, p, w)$ at the price vector $p^i$ when the reference price is $\bar{p}$. For a fixed $\bar{p}$ and viewed as a function of $(p, w)$, $e(p, p^i, V(p, p, w))$ is also an indirect utility function. By definition, $e$ is expressed in terms of money, hence, it is known as a money metric indirect utility function. Thus, the dollar value of the welfare change for the consumer is

$$e(p, p, V(p, p, w)) - e(p, p, V(p, p, w)).$$

(5.2)

Since $e(p, p^i, V(p, p, w))$ is increasing in $V(p, p, w)$ (see Proposition 6), the sign of the expression in (5.2) is identical to the sign of $V(p, p, w) - V(p, p^0, w)$, which is what we require to determine from (5.1). Suppose that we consider only two goods. We vary only the price of good 1 and normalize the price of good 2 by

$$p_1^0 = p_1^1 = \bar{p}_2 = 1.$$

Thus, in $(x_2, x_1)$ space, the budget line $p_1 x_1 + x_2 = w$ is pivoted on the vertical axis by $w = e(p, p, U)$ where $U = V(p, p, w)$. In order to operationalize (5.2), we need to specify the choice of $p^i$. Two choices have been made in the literature: the initial price, $p^0$, and the final price, $p^1$. When $p^i = p^0$, the resulting measure of welfare change is known as equivalent variation (EV) and when $p^i = p^1$, the resulting measure of welfare change is known as compensating variation (CV).

Let $U_0 = V(p, p^0, w)$ and $U_1 = V(p, p^1, w)$. It follows that $w = e(p, p^0, U_0)$ and $w = e(p, p^1, U_1)$. Thus, successively setting $p^i = p^0$ and $p^i = p^1$ in (5.2): 

$$EV : e(p, p^0, U_1) - e(p, p^0, U_0) = e(p, p^0, U_1) - w.$$ 

(5.3)

$$CV : e(p, p^1, U_1) - e(p, p^1, U_0) = w - e(p, p^1, U_0).$$

EV is the extra dollar income (positive or negative) that if given at the initial prices to the consumer, makes the consumer indifferent to the price change. CV repeats the same thought experiment but replaces initial prices by final prices. The dollar amounts calculated based on the two measures will typically be different. However, both measures of welfare give an identical welfare ranking in the sense that if the consumer is better off (worse off) after the price change using one measure, then the consumer is also better off (worse off) using the other measure.
Figure 5.1: The case of a fall in price of good 1. The original price, $p^0$, is the reference price.

5.2. A diagrammatic exposition of equivalent and compensating variation

We illustrate the basic concepts first through a series of diagrams, assuming that the initial price is the reference price that stays fixed during the analysis. The analysis can equally well be conducted if the initial price is not the reference price.

I. Fall in Price (Original price is the reference price)

Figure 5.1 shows the case of a fall in price. Suppose that the original price $p^0$ equals the reference price $\bar{p} = (p^0_1, 1)$; thus, there is no gain-loss utility and the solutions with and without gain-loss utility coincide. The original budget constraint is shown by the line $B_{p^0,w}$ and the original equilibrium is shown at the point $A$.

Consider now a fall in the price of good 1 so that the new price vector is $p^1 = (p^1_1, 1)$ such that $p^1_1 < p^0_1$; thus, the price of good 1 falls. This pushes the consumer into the domain of gains where the indifference curves are relatively steeper (relatively higher marginal utility of good 1). The new budget constraint is shown by the line $B_{p^1,w}$.

1. In the presence of gain-loss utility, the new equilibrium is shown at $B$, and the associated utility is indicated by $U^1(\succeq_{p^0,p^1})$ (the reference price stays fixed at $\bar{p} = p^0 = (p^0_1, 1)$).

2. In the classical case, in the absence of gain-loss utility, at the new prices $p^1$, the equilibrium is shown at the point $C$ and the utility at the new equilibrium is shown next to the relevant indifference curve as $U^1(\succeq)$. There is no presumption that $U^1(\succeq_{p^0,p^1})$ equals
The EV is the dollar amount that if given to the consumer at the original price vector \( p^0 \) would make the consumer indifferent to accepting the new price vector \( p^1 \). This takes the form of starting at the original price ratio (in this case \(-p^1_0/1\)) and shifting the budget constraint outwards, in a parallel manner, so that the new budget constraint is just tangent to the final indifference curve which is either \( U^1(\succeq_{p^0,p^1}) \) or \( U^1(\succeq) \) depending on whether we allow for gain-loss utility or not. Figure 5.1 illustrates the two cases separately.

1. Absence of gain-loss utility: In the classical case, the relevant new indifference curve is \( U^1(\succeq) \) and the budget constraint, shown as the line \( EE \), is just tangent to it at the point \( C' \). This budget constraint hits the vertical axis at point \( E \) and corresponds to an expenditure level \( w' = e(p^0,U^1(\succeq)) \) (recall that \( p^0_2 = p^1_2 = 1 \)). EV is then given by the vertical distance

\[
e(p^0,U^1(\succeq)) - w = w' - w > 0.
\]

2. Presence of gain-loss utility: It might be tempting to construct an argument along the following lines: The relevant new indifference curve after a fall in price is \( U^1(\succeq_{p^0,p^1}) \) and the budget constraint reflecting a price ratio of \(-p^0_2/p^0_1\) (shown as the line \( DD \)) is just tangent to it at the point \( B' \). This budget constraint hits the vertical axis at the expenditure level \( e' \), so EV is given by the vertical distance

\[
e' - w > 0.
\]

However, there is a flaw in this argument. The consumption bundle \( B' \) gives a level of utility \( U^1 \) when the reference price is \( p^0 \) and the actual price is \( p^1 \), i.e., \( U^1(\succeq_{p^0,p^1}) \). However, we seek a level of utility \( U^1 \) when the actual and reference price is \( p^0 \), i.e., \( U^1(\succeq_{p^0,p^0}) \). Let \( \hat{U}(\succeq_{p^0,p^0}) \) be the utility corresponding to the consumption bundle \( B' \) when the actual and reference price is \( p^0 \). At \( B' \) we have \( \hat{U}(\succeq_{p^0,p^0}) < U^1(\succeq_{p^0,p^1}) \); these are labelled in the diagram. To see this, let \( x \) be the consumption bundle corresponding to point \( B' \). Then the utility at point \( B' \) on the two different indifference curves at prices \( p^0 \) and \( p^1 \) when the reference price is \( p^0 \) is

\[
\begin{align*}
p^0 : & \quad \hat{U}(\succeq_{p^0,p^0}) = u(x) \\
p^1 : & \quad U^1(\succeq_{p^0,p^1}) = u(x) + \mu(p^0 - p^1)x_1 > \hat{U}(\succeq_{p^0,p^0}).
\end{align*}
\]

We assume that preferences are monotonic. Hence, starting from \( B' \), in order to increase the level of utility beyond \( \hat{U} \) to \( U^1(\succeq_{p^0,p^0}) \) the relevant consumption bundle must lie to the north-east of point \( B' \). This is shown at point \( G \). The relevant budget constraint with slope \(-p^0_2/p^0_1\) is shown by \( FF \) and the corresponding expenditure by \( e(p^0,p^0,U^1(\succeq_{p^0,p^0})) \). Thus, the correct measure of equivalent variation under gain-loss utility is

\[
EV = e(p^0,p^0,U^1(\succeq_{p^0,p^0})) - w > e' - w > e(p^0,U^1(\succeq)) - w.
\]
Comparing the two cases, the classical case leads to a underestimation of the welfare change of a price fall by the vertical distance $FE$ or equivalently by the monetary amount

$$e \left( p^0, p^0, U^1 (\geq p^0, p^0) \right) - e \left( p^0, U^1 (\geq) \right) > 0.$$ 

The reason is that under gain-loss utility, a price decrease relative to the reference price gives rise to extra utility over and above that revealed by classical preferences. Hence, the consumer needs extra income to state indifference to a fall in price or an increase in income. We may decompose the total price effect as follows:

$$\begin{align*}
A \rightarrow B' & \quad \text{Classical Income effect} \\
B' \rightarrow G & \quad \text{Reference income effect} \\
G \rightarrow B & \quad \text{Substitution effect}
\end{align*}$$

5.3. Formulas for equivalent and compensating variation

Continue to assume that the price of good 2 does not change and only the price of good 1 changes from $p^0_1$ to $p^1_1$ (we do not specify whether this is an increase or decrease). The price of good 2 is normalized to $p^0_2 = p^1_2 = 1$. Thus, the initial price is $p^0 = (p^0_1, 1)$ and the final price is $p^1 = (p^1_1, 1)$. From (5.3) we have $EV = e(\bar{p}, p^0, U^1) - w$ and $w = e(\bar{p}, p^1, U^1)$. Thus,

$$EV (\bar{p}, p^0, p^1, w) = e(\bar{p}, p^0, U^1) - e(\bar{p}, p^1, U^1).$$

From Proposition 9 we have $\frac{\partial e(\bar{p}, p, U)}{\partial p_1} = h^T_1(\bar{p}, p, U)$ where $h^T_1(\bar{p}, p, U)$ is given by (4.34). Thus, we can write $e(\bar{p}, p^0, U) = \int_0^{p^0_1} h^T_1(\bar{p}, p, U) dp$ (recall that price of good 2 is fixed). Rewriting (5.5):

$$EV = \int_0^{p^0_1} h^T_1 dp - \int_0^{p^1_1} h^T_1 dp = \int_0^{p^0_1} h^T_1 dp - \left[ \int_0^{p^0_1} h^T_1 dp + \int_{p^0_1}^{p^1_1} h^T_1 dp \right]$$

$$\Rightarrow EV = \begin{cases} 
\int_0^{p^0_1} h^T_1 (\bar{p}, p, U^1) dp > 0 & \text{if } p^1_1 < p^0_1 \\
-\int_{p^0_1}^{p^1_1} h^T_1 (\bar{p}, p, U^1) dp < 0 & \text{if } p^1_1 > p^0_1 
\end{cases},$$

where $h^T_1$ is defined in (4.34).  

Using the properties of $h^T_1$ in Section 4.9, we depict $EV$ in Figure 5.2. The classical measure of EV is given by the area under the Hicksian demand curve, i.e., the area ABCDE. However, the correct behavioral measure is given by the area under the transformed Hicksian curve as indicated in (5.6). This is the shaded region in Figure 5.2; from (4.37) this area is greater than the classical measure of area under the Hicksian curve.

\footnote{In the second row of (5.6) we may also use the convention of taking absolute values to ensure that $EV > 0$, which enables us to compare the actual values of $EV$ in the domain of gains and losses.}
**Proposition 12**: Let \( U : X \times \mathbb{R}^2_+ \times \mathbb{R}^2_+ \rightarrow \mathbb{R} \) represent continuous, locally non-satiated and strictly convex preferences, \( \succeq_{\bar{p}, p} \). Using the Hicksian demand, \( h_1(\bar{p}, p, U^1) \), will lead to a lower prediction of the absolute value of EV relative to using the transformed Hicksian demand, \( h_1^T(\bar{p}, p, U^1) \).

Proof: We have \( 1 - \mu \lambda \frac{\partial g_1(\bar{p}, p_1)}{\partial p_1} > 0 \), thus, from (4.34), (5.6) we get

\[
\begin{cases}
\int_{p_1^0}^{p_1^1} h_1^T(\bar{p}, p, U^0)dp_1 = \int_{p_1^0}^{p_1^1} h_1(\bar{p}, p, U^1)dp_1 > \int_{p_1^0}^{p_1^1} h_1(\bar{p}, p, U^0)dp_1 \quad \text{if } p_1^0 > p_1^1.
\end{cases}
\]
(5.7)

From (5.7) we get the required result. 

From (5.3), we have \( CV = w - e(\bar{p}, p^1, U^0) \) and \( w = e(\bar{p}, p^0, U^0) \). Thus, \( CV(\bar{p}, p^0, p^1, w) = e(\bar{p}, p^0, U^0) - e(\bar{p}, p^1, U^0) \). Thus,

\[
CV(\bar{p}, p^0, p^1, w) = e(\bar{p}, p^0, U^0) - e(\bar{p}, p^1, U^0).
\]
(5.8)

The expression for \( CV \) that is analogous to (5.6) is given by

\[
CV(\bar{p}, p^0, p^1, w) = \int_{p_1^0}^{p_1^1} \left( 1 - \mu \lambda \frac{\partial g_1(\bar{p}, p_1)}{\partial p_1} \right) h_1(\bar{p}, p, U^0)dp.
\]
(5.9)

Since \( h_1(\bar{p}, p, U^0) < h_1(\bar{p}, p, U^1) \) it follows that \( CV(\bar{p}, p^0, p^1, w) < EV(\bar{p}, p^0, p^1, w) \).
5.4. Deadweight loss from commodity taxation

In this section, we calculate the deadweight loss from commodity taxation under reference dependence. We hold fixed the price of good 2 at $p_2^0 = p_2^1 = 1 = \bar{p}_2$. Suppose that the initial price of good 1 is $p_1^0$, so the initial price vector is $p^0 = (p_1^0, 1)$. The government then levies a commodity tax $\tau > 0$ per unit on the purchase of good 1 so that the final price of good 1 is $p_1^1 = p_1^0 + \tau$. The tax paid by the consumer equals $\tau x_1(\bar{p}, p^1, w)$, where $p^1 = (p_1^0 + \tau, 1)$.

The calculation $EV(\bar{p}, p^0, p^1, w) = e(\bar{p}, p^0, U^1) - e(\bar{p}, p^1, U^1)$ tells us how much money the consumer is willing to pay in dollar terms that leaves him indifferent to facing the price change from $p_1^0$ to $p_1^0 + \tau$, while utility is held fixed at $U^1$. This is a measure of the welfare change (negative in this case) from commodity taxation. We would like to compare this welfare change with the welfare change that arises from levying a lumpsum tax $T$ on the consumer that raises the same amount of tax revenue as under commodity taxation, so

$$T = \tau x_1(\bar{p}, p^1, w).$$

Since $x_1(\bar{p}, p^1, w) = h_1(\bar{p}, p^1, U^1)$, where $U^1 = V(\bar{p}, p^1, w)$, we can rewrite $T = \tau h_1(\bar{p}, p^1, U^1)$. Since $EV$ is a dollar measure of welfare change, and it is negative for the case of an increase in price following the commodity tax, the consumer is worse off under commodity taxation if $-EV = |EV| > T$; lumpsum taxation leaves the consumer with greater wealth in this case. The expression $-EV - T$ is known as the deadweight loss of taxation, $DWL$. Substituting for $EV$ and $T$ we get

$$DWL = e(\bar{p}, p^1, U^1) - e(\bar{p}, p^0, U^1) - \tau h_1(\bar{p}, p^1, U^1); \ (p_1^1 = p_1^0 + \tau > p_1^0). \quad (5.10)$$

Using (5.6) and noting that we have $-EV$ in this case, we can rewrite (5.10) as

$$DWL = \int_{p_1^0}^{p_1^1 + \tau} \left(1 - \mu \lambda \frac{\partial g_1(\bar{p}, p_1)}{\partial p_1} \right) h_1(\bar{p}, p, U^1) dp_1 - \tau h_1(\bar{p}, p^1, U^1). \quad (5.11)$$

A simple calculation shows that

$$\int_{p_1^0}^{p_1^1 + \tau} h_1(\bar{p}, p_1^0 + \tau, U^1) dp_1 = h_1(\bar{p}, p_1^0 + \tau, U^1) \int_{p_1^0}^{p_1^1 + \tau} dp = \tau h_1(\bar{p}, p_1^0 + \tau, U^1) \quad (5.12)$$

Substituting (5.12) in (5.11) we get

$$DWL = \int_{p_1^0}^{p_1^1 + \tau} \left[ h_1(\bar{p}, p, U^1) - h_1(\bar{p}, p_1^0 + \tau, U^1) \right] dp_1 - \mu \lambda \int_{p_1^0}^{p_1^1 + \tau} \frac{\partial g_1(\bar{p}, p_1)}{\partial p_1} h_1(\bar{p}, p, U^1) dp_1. \quad (5.13)$$

The first term is the classical measure of $DWL$ (albeit taking gain-loss utility into account) and the second term which is positive ($\frac{\partial g_1(\bar{p}, p_1)}{\partial p_1} \in (-1, -\theta)$ and $\lambda \geq 0$) is a correction to the classical measure of $DWL$. The interpretation of the second term is simple. Recall the
Figure 5.3: Deadweight loss in the presence of gain-loss utility.

Figure 5.4: Deadweight loss in the presence of gain-loss utility.
interpretation of $-\mu \lambda \frac{\partial h_i(p, \overline{p})}{\partial p_i} h_i$ following Definition 5 as the shadow cost of a unit increase in prices that works through the gain-loss channel for $h_i$ units of demand. The second term integrates over all the relevant shadow costs over the interval $[p_1^0, p_1^0 + \tau]$.

By contrast, the classical measure of deadweight loss is given by

$$DWL_{\mu=0} = \int_{p_1^0}^{p_1^0 + \tau} [h_1(p, U^1) - h_1(p_1^0 + \tau, U^1)] \, dp_1.$$  \hfill (5.14)

We now compare the two measures of deadweight loss in the absence and in the presence of gain-loss utility, $DWL_{\mu=0}$ given in (5.14) and $DWL_{\mu>0}$ given in (5.13).

(i) In Figure 5.3 there is no gain-loss utility ($\mu = 0$). The original equilibrium at the price $p_1^0$ is at $A$, utility is $U^0 (\geq )$, and the final equilibrium at a price $p_1^1 = p_1^0 + \tau$ is at $B$, giving rise to utility $U^1 (\geq )$. Passing a line through $B$ at the price $p_1^0$ gives the income needed to consume the consumption bundle $B$ if a lumpsum tax, $T$, were available. Since $p_2$ is normalized to 1, the height of point $D$ gives the required expenditure and $T$ equals the vertical distance $wD$. However, the distortionary commodity tax, $\tau$, allows for a utility of $U^1 (\geq )$ at point $B'$ but the income is even lower. $DWL_{\mu=0}$ is given by the vertical distance $DE$.

(ii) The corresponding diagram in the presence of gain-loss utility ($\mu > 0$) is shown in Figure 5.4. We assume that the original price vector $(p_1^0, 1)$ is also the reference price vector, $\overline{p}$. The utility levels $U^0$ and $U^1$ are now indexed by the reference price and the actual price; hence, these are written as $U^0 (\geq_{p^0, \overline{p}})$ and $U^1 (\geq_{p^0, \overline{p}})$. The budget line with slope $-\frac{p_2^0}{p_1^0}$ that is tangent at point $B'$ no longer gives the required utility level $U^1$ at the price vector $p^0$. Suppose that the consumption bundle $x'$ corresponds to point $B'$. At a price $p^1 > p^0$, the consumption bundle at point $B'$ gives the utility $u(x') - \theta \mu (p_1^1 - p_1^0) = U^1 (\geq_{p^0, \overline{p}})$. But at a price $p^0$, the relevant preferences are $\geq_{p^0, \overline{p}}$ and the utility corresponding to the consumption bundle $x'$ at point $B'$ is $u(x') > U^1 (\geq_{p^0, \overline{p}})$. Given monotonicity of preferences, we seek another consumption bundle $x$, $x' \succ x$, which lowers the utility starting from $u(x')$ to the level $U^1 (\geq_{p^0, \overline{p}})$. Such a consumption bundle is shown at the point $C'$ and the corresponding utility is at the required level $U^1 (\geq_{p^0, \overline{p}})$. The deadweight loss is now given by the much greater distance $DE$ relative to that shown in Figure 5.3.

Example 14 : Consider the parameter values in Example 13 when only the price of good 1 varied: $\overline{U} = 6$, $\overline{p}_1 = 2$, $\overline{p}_2 = p_2 = 1$, $\mu = 0.2$, $\theta = 2.5$. Let the tax rate $\tau = 0.20$ and consider two levels of initial prices:

(i) $p_1^0 = 1.5 < \overline{p}_1$ and $p_1^0 + \tau = 1.7 < \overline{p}_1$. In this case we have

$$DWL_{\mu=0} = 0.017678, \quad DWL_{\mu>0} = 0.2068, \quad \frac{DWL_{\mu>0}}{DWL_{\mu=0}} = 11.698.$$ 

(ii) $p_1^0 = 3.0 > \overline{p}_1$ and $p_1^0 + \tau = 3.2 > \overline{p}_1$. In this case we have

$$DWL_{\mu=0} = 0.017678, \quad DWL_{\mu>0} = 0.34121, \quad \frac{DWL_{\mu>0}}{DWL_{\mu=0}} = 19.301.$$
Thus, the classical analysis severely underestimates DWL of taxation. For the chosen parameter values, the DWL is actually 11.698 times higher in the domain of gains and 19.301 times higher in the domain of losses.

References


